**About the Author** – Kirk Weiler has been a teacher of mathematics at Arlington High School for the past 16 years. For the past four years he has also served as the Math Department Coordinator. While at Arlington, he has taught courses ranging from Algebra 1 to Advanced Placement Calculus. He was educated as an engineer, earning both bachelors and masters degrees in engineering from the University of Illinois and Cornell University, respectively. He then left engineering for education, earning his masters in mathematics education from Syracuse University. In 2006 he earned his National Board Certification in Young and Adult Mathematics Education. From 2006 until 2008, he served as the Editor-in-chief of the Arlington Algebra Project, a collaborative effort by 26 middle school and high school teachers to write an electronic textbook for an introductory Algebra 1 course. In 2008, Kirk founded eMathInstruction and published Algebra 2 with Trigonometry. Common Core Algebra II is eMathInstruction’s third offering.

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COMMON CORE ALGEBRA II

VERSION 1.0

BY KIRK WEILER

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ALGEBRAIC ESSENTIALS REVIEW

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Lesson #2 – Solving Linear Equations
Lesson #3 – Common Algebraic Expressions
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**COMMON CORE ALGEBRA II, UNIT #1 – ESSENTIAL ALGEBRA CONCEPTS – LESSON #1**

**NAME:** ____________________________________  **DATE:** __________________

**VARIABLES, TERMS, AND EXPRESSIONS**

**COMMON CORE ALGEBRA II**

Mathematics has developed a language all to itself in order to clarify concepts and remove ambiguity from the analysis of problems. To achieve this, though, we have to agree on basic definitions so that we can all speak this same language. So, we start our course in Algebra II with some basic review of concepts that you saw in Algebra I.

**SOME BASIC DEFINITIONS**

**Variable:** A quantity that is represented by a letter or symbol that is unknown, unspecified, or can change within the context of a problem.

**Terms:** A single number or combination of numbers and variables using exclusively multiplication or division. This definition will expand when we introduce higher-level functions.

**Expression:** A combination of terms using addition and subtraction.

**Exercise #1:** Consider the expression $2x^2 + 3x - 7$.

(a) How many terms does this expression contain?  
(b) Evaluate this expression, without your calculator, when $x = -3$. Show your calculations.

(c) What is the sum of this expression with the expression $5x^2 - 12x + 2$?

**Exercise #2:** Most students learn that to add two like terms they simply add the coefficients and leave the variables and powers unchanged. But, why does this work? Below is an example of the technical steps to combine two like terms. What real number property justifies the first step?

$$4x^2y + 6x^2y = x^2y(4 + 6) \quad \text{Justification?}$$

$$= x^2y(10) = 10x^2y$$

**LIKE TERMS**

**Like Terms:** Two or more terms that have the same variables raised to the same powers. In like terms, only the coefficients (the multiplying numbers) can differ.

**Exercise #2:** Most students learn that to add two like terms they simply add the coefficients and leave the variables and powers unchanged. But, why does this work? Below is an example of the technical steps to combine two like terms. What real number property justifies the first step?

$$4x^2y + 6x^2y = x^2y(4 + 6) \quad \text{Justification?}$$

$$= x^2y(10) = 10x^2y$$
**Exercise #3:** The procedure for simplifying the linear expression $8(2x+3)+5(3x+1)$ is shown below. State the real number property that justifies each step.

$8(2x+3)+5(3x+1)=8\cdot2x+8\cdot3+5\cdot3x+5\cdot1$  

$=8\cdot2x+24+(5\cdot3)x+5=16x+24+15x+5$  

$=16x+15x+24+5$  

$=x(16+15)+24+5$  

$=31x+(24+5)$  

$=31x+29$  

**Exercise #4:** Because we used real number properties to transform the expression $8(2x+3)+5(3x+1)$ into a simpler form $31x+29$, these two expressions are **equivalent**. How can you test this equivalency? Show work for your test.
1. For each of the following expressions, state the number of terms.

(a) $3x^2 - 1$  
(b) $8x + 7x^2 - 2 + x^3$  
(c) $7xy - 2x^2 y^2 + \frac{1}{2} xy^4$

2. Simplify each of the following expressions by combining like terms. Be careful to only combine terms that have the same variables and powers.

(a) $2x^2 + 8x - 1 + 5x^2 - 2x - 8$  
(b) $-5x^2 - 2x + 10 - x^2 + 7x + 5$

(c) $4x^2 y - 2xy^2 + 9xy^2 - x^2 y$  
(d) $7x^2 - 2x^2 y + 4xy^2 - y^3 + 2x^2 + 9x^2 y + 4y^3$

3. Given the algebraic expression $\frac{12x + 12}{x^2 - 1}$ do the following:

(a) Evaluate the expression for when $x = 7$.  
(b) Evaluate the expression for when $x = 4$.

(c) Nina believes that this expression is equivalent to dividing 12 by one less than $x$. Do your results from (a) and (b) support this assertion? Explain.
4. Classify each of the following as either a monomial (single term), a binomial (two terms) or a trinomial (three terms).

(a) $4x^2$       (b) $-3x^2 + 2x - 1$       (c) $16 - x^2$

(d) $x^2y^2 + 25$       (e) $\frac{5x^3}{3}$       (f) $16 + 10t - 4t^2$

5. Use the distributive property first and then combine each of the following linear expressions into a single, equivalent binomial expression.

(a) $5(2x + 3) + 2(4x - 1)$

(b) $2(10x + 1) - 3(4x - 5)$

6. Which of the following is equivalent to the expression $2(x - 6) + 4(2x + 1) + 3$?

(1) $8(x - 2)$

(2) $5(2x - 1)$

(3) $4(2x + 3)$

(4) $10(x - 1)$

7. Each step in simplifying the expressions you worked with in 5 and 6 can be justified using one of the major properties of real numbers reviewed in the lesson. Justify each step below with either the commutative, associative or distributive properties when simplifying the expression $8(3x + 1) + 2(5x + 7)$.

$8(3x + 1) + 2(5x + 7) = 24x + 8 + 10x + 14$

$= 24x + 10x + 8 + 14$

$= (24x + 10x) + (8 + 14)$

$= x(24 + 10) + 22$

$= 34x + 22$
We will learn many new equation solving techniques in Algebra II, but the most basic of all equations are those where the variable, say $x$, is only raised to the first power. These are known as **linear equations**. You need to have good fluency with solving these equations in order to be successful in the beginning portions of Algebra II. Let's start with some practice.

**Exercise #1:** Solve each of the following linear equations for the value of $x$.

(a) $3x + 5 = 26$

(b) $8x - 7 = 4x - 5$

(c) $\frac{x + 8}{2} = -6$

(d) $6(x + 4) - 2(x - 1) = 2x + 20$

It is important to understand that each step in solving one of these equations can be justified by either using one of the properties of real numbers (from the last lesson) or a property of equality (such as the additive or multiplicative properties).

**Exercise #2:** Justify each step in solving $2(x + 7) + 4x = 44$ using either a property of real numbers (commutative, associative, or distributive) or a property of equality (additive or multiplicative).

\[
\begin{align*}
2(x + 7) + 4x &= 44 \\
2x + 14 + 4x &= 44 \\
2x + 4x + 14 &= 44 \\
x(2 + 4) + 14 &= 44 \\
x(6) + 14 &= 44 \\
x(6) &= 30 \\
\frac{6x}{6} &= \frac{30}{6} \\
x &= 5
\end{align*}
\]
Strange things can sometimes happen when solving linear (and other) equations. Sometimes we get no solutions at all, in which case the equation is known as **inconsistent**. Other times, any value of $x$ will solve the equation, in which case it is known as an **identity**.

**Exercise #3:** Try to solve the following equation. State whether the equation is an **identity** or **inconsistent**. Explain.

$$6x - 2(x + 4) = 3(x + 2) + x - 5$$

**Exercise #4:** An identity is an equation that is true for all values of the substitution variable. Trying to solve them can lead to confusing situations. Consider the equation:

$$2x - 6 + x - 1 = 3(x - 3) + 2$$

(a) Test the values of $x = 5$ and $x = 3$ in this equation. Show that they are both solutions.

(b) Attempt to solve the equation until you are sure this is an identity.

**Exercise #5:** Which of the following equations are identities, which are inconsistent, and which are neither?

(a) $8x - 2(x + 3) = 5(x - 1) + x$

(b) $\frac{4x + 2}{2} + 8 = 2x + 9$

(c) $2x + 8 - (x - 7) = 2(2x - 3)$

(d) $2x + 1 + 2(x - 1) = \frac{16x - 4}{4}$
SOLVING LINEAR EQUATIONS
COMMON CORE ALGEBRA II HOMEWORK

**FLUENCY**

1. Solve each of the following linear equations. If the equation is inconsistent, state so. If the equation is an identity, also state so. Reduce any non-integer answers to fractions in simplest form.

(a) \(7x + 5 = 2x - 35\)  
(b) \(\frac{x}{3} - 7 = -5\)  
(c) \(4x + 5 = 4x - 1\)

(d) \(\frac{5(x - 3)}{2} - 1 = 14\)  
(d) \(3(x - 1) + 2 = x + 9\)  
(e) \(4x - (2x - 1) = x + 5 + x - 6\)

(f) \(5(2x - 6) + 2(4x + 3) = 8x - 9\)  
(g) \(\frac{2x + 5}{6} = \frac{x}{18}\) (Cross multiply to begin)

(h) \(\frac{10x - 4}{2} + 7 = 5(x + 1)\)  
(i) \(18 - 2(x + 7) = \frac{8x - 20}{2} - 2\)
APPLICATIONS

2. Laura is thinking of a number such that the sum of the number and five times two more than the number is 26 more than four times the number. Determine the number Laura is thinking of.

3. As if #2 wasn't confusing enough, Laura is now trying to come up with a number where three less than 8 times the number is equal to half of 16 times the number after it was increased by 1. She can't seem to find a number that works. Explain why.

4. When finding the intersection of two lines from both Algebra I and Geometry, you first "set the linear equations equal" to each other. Find the intersection point of the two lines whose equations are shown below. Be sure to find both the x and y coordinates.

\[ y = 5x + 1 \text{ and } y = 2x - 11 \]

REASONING

5. Explain why you cannot find the intersection points of the two lines shown below. Give both an algebraic reason and a graphical reason.

\[ y = 4x + 1 \text{ and } y = 4x + 10 \]
In Algebra II we will spend a lot of time evaluating and simplifying algebraic expressions. Just to be clear:

**Algebraic Expression**

**Algebraic expressions** are just combinations of constants and variables using the typical operations of addition, subtraction, multiplication, and division along with exponents and roots (square roots, cube roots, etcetera).

It is important to be able to evaluate algebraic expressions for values of the variables contained in them.

**Exercise #1:** Consider the algebraic expression $4x^2 + 1$.

(a) Describe the operations occurring within this expression and the order in which they occur.

(b) Evaluate this expression for the replacement value $x = -3$. Show each step in your calculation. Do not use a calculator.

**Exercise #2:** Consider the more complex algebraic expression (known as a rational expression) $\frac{4x + 3}{x^3 - 7}$.

(a) Without using your calculator, find the value of this expression when $x = 3$. Reduce your answer to simplest terms. Show your steps.

(b) If a student entered the following expression into their calculator, it would give them the incorrect answer. Why?

$$4(3) + 3 / 3^3 - 7$$

Expressions can contain more complex operators, such as the square and cube roots as well as the absolute value. We will need each of these over the span of this course, so some practice with all of them is warranted.

**Exercise #3:** Is the absolute value expression $|x - 8| + 2$ equivalent to $|x| + 10$? How can you check this?
**Exercise #4:** Consider the algebraic expression $\sqrt{25-x^2}$, which contains a square root.

(a) Evaluate this expression for $x = -3$.  
(b) Why can you *not* evaluate the expression for $x = 13$?

(c) Max thinks that the square root operation distributes over the subtraction. In other words, he believes the following equation is an identity:

$$\sqrt{25-x^2} = 5 - x$$

Show that this is *not* an identity.

Algebraic expressions can become quite complicated, but if you consider *order of operations* and work generally from *inside to outside* then you can evaluate any expression for replacement values.

**Exercise #5:** Consider the rather complicated expression $\sqrt{|x-8|}$.

(a) What operation comes last in this expression?  
(b) Evaluate the expression for $x = 2$. Simplify it completely.

**Exercise #6:** Which of the following is the value of $\frac{\sqrt{4x+9-x^2}}{3}$ when $x = 10$?

(1) 31  
(2) 24  
(3) 18  
(4) 84
1. Which of the following expressions has the greatest value when \( x = 5 \)? Show how you arrived at your choice.

\[
2x^2 + 7 \quad \frac{x^3 - 5}{3} \quad \frac{10x - 2}{x - 3}
\]

2. A **zero** of an expression is a value of the input variable that results in the expression having a value of zero (catchy and appropriate name). Is \( x = 3 \) a zero of the quadratic expression shown below? Justify your yes/no answer.

\[4x^2 - 8x - 12\]

3. Which of the following is the value of the rational expression \( \frac{2 - 3x^2}{6x + 4} \) when \( x = -2 \)?

(1) \(-2\frac{1}{2}\)  
(2) \(-\frac{5}{8}\)  
(3) \(1\frac{1}{2}\)  
(4) \(\frac{2}{7}\)

4. If \( x = 5 \) and \( y = -2 \) then \( \frac{x + y}{x^2 - y^2} \) is

(1) \(\frac{1}{7}\)  
(2) \(\frac{13}{3}\)  
(3) \(\frac{3}{29}\)  
(4) \(\frac{7}{19}\)
5. What is the value of \(|x - 10| - |x + 3|\) if \(x = 2\)?

(1) 7 \hspace{1cm} (3) 3
(2) 5 \hspace{1cm} (4) 17

6. If \(x = 2\) then \(\frac{\sqrt{4x^2 + 2x + 5}}{10}\) has a value of

(1) \(\frac{5}{2}\) \hspace{1cm} (3) \(\frac{2}{5}\)
(2) \(\frac{7}{5}\) \hspace{1cm} (4) \(\frac{1}{2}\)

APPLICATIONS

7. The revenue, in dollars, that eMathInstruction makes off its videos in a given day depends on how many views they receive. If \(x\) represents the number of views, in hundreds, then the profit can be found with the expression:

\[ \frac{1}{2} x^2 + 6x - 10 \]

How much revenue would they make if their videos were viewed 600 times?

REASONING

8. Sameer believes that the two expressions below are equivalent. Test values and see if you can build evidence for or against this belief.

\((x - 3)(x + 8)\) \hspace{1cm} \(x^2 + 5x - 24\)
Exponents, at their most basic, represent \textit{repeated multiplication}. The way they combine, or don't combine, is dictated by this simple premise.

\textbf{Exercise \#1:} The following four steps are given to find the product of the monomials \(-2x^5\) and \(4x^2\).

\[
(-2x^5) \cdot (4x^2)
\]

(1) \(-2 \cdot (x^5 \cdot 4) \cdot x^2\)

(2) \(-2 \cdot (4 \cdot x^5) \cdot x^2\)

(3) \((-2 \cdot 4) \cdot (x^5 \cdot x^2)\)

(4) \(-8x^7\)

Students (and teachers) can forget the basic properties used in simplifying the product of two monomials because we tend to pick up on the pattern of multiplying the numerical coefficients and adding the powers without thinking about the commutative and associative properties that justify our manipulations.

\textbf{Exercise \#2:} Find the product of each of the following monomials.

(a) \((5x^2)(3x^6)\) \hspace{1cm} (b) \((-2x)(-6x^4)\) \hspace{1cm} (c) \(\left(\frac{3}{2}x^4\right)(6x^{10})\) \hspace{1cm} (d) \(\left(4x^3\right)^2\)

Remember, monomials (or terms) can have more than one variable, just as long as they are all combined using multiplication and division only. Multiplying monomials that contain more than one variable still just involves application of exponent laws and repeated use of the associative and commutative properties.

\textbf{Exercise \#3:} Find each of the following products, which involve monomials of multiple variables. Carefully consider what you are doing before applying patterns.

(a) \((4x^3y^2)(5xy^5)\) \hspace{1cm} (b) \((-2x^7y^3)(-4x^2y^6)\) \hspace{1cm} (c) \(\left(\frac{1}{2}xy\right)\left(\frac{5}{2}x^2y^5\right)\)
One of the key skills we will need this year will be factoring expressions, especially factoring out a common factor. To build some skills with this, consider the following problem.

**Exercise #4:** Fill in the missing blank in each of the following equations involving a product such that the equation is then an identity.

(a) \(6x^5 = (2x^2)(\underline{\quad})\)  
(b) \(12x^8 = (4x^3)(\underline{\quad})\)  
(c) \(20x^2y^4 = (-2xy^3)(\underline{\quad})\)

The final skill we will review in this lesson is using the **distributive property of multiplication (and division)** over **addition (and subtraction).**

**Exercise #5:** Use the distributive property to multiply the following monomials and polynomials.

(a) \(2x(5x + 3)\)  
(b) \(5x^3(2x^2 - 3x + 6)\)  
(c) \(-7x^2(x^2 - 2x + 3)\)

(d) \(xy^2(x^2 - y^2)\)  
(e) \(3x^2y^4(2x^2y + xy^2 - 4y^3)\)

Now, to build our way up to **factoring** in later units, let's make sure we can fill in missing portions of products.

**Exercise #6:** Similar to Exercise #4, fill in the missing portion of each product so that the equation is an identity.

(a) \(8x^2 - 12x = 4x(\underline{\quad})\)  
(b) \(7x^4 - 21x^3 - 28x^2 = 7x^2(\underline{\quad})\)

(c) \(10x^3y^2 - 20x^2y^3 + 35xy^5 = 5xy^2(\underline{\quad})\)

(d) \(4x^2(x - 2) - 9(x - 2) = (x - 2)(\underline{\quad})\)
BASIC EXPONENT PROPERTIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. The steps in finding the product of \( (3x^2y^4) \) and \( (7x^5y^2) \) are shown below. Fill in either the associative property or the commutative property to justify each step.

\[
\begin{align*}
(3x^2y^4)(7x^5y^2) &= (3x^2)(y^4 \cdot 7)(x^5y^2) \\
&= (3x^2)(7y^4)(x^5y^2) \\
&= 3(x^2 \cdot 7)(y^4x^5y^2) \\
&= 3(7x^2)(x^5y^2) \\
&= (3 \cdot 7)(x^3x^5)(y^4y^2) \\
&= 21x^7y^6
\end{align*}
\]

2. Find each of the following products of monomials.

(a) \( (3x^2)(10x^4) \)  
(b) \( (-2x^5)(-9x) \)  
(c) \( (4x^2y)(8x^5y^3) \)  
(d) \( (5x^4)^2 \)

(e) \( (-4t^2)(-15t^5) \)  
(f) \( (7x)(5xy^4) \)  
(g) \( \left( \frac{2}{3}x^4 \right)(12x) \)  
(h) \( (2x^2)(5x)(-6x^4) \)

3. Fill in the missing portion of each product to make the equation an identity.

(a) \( 18x^6 = 3x^2(\quad) \)  
(b) \( 40x^2y^7 = 8xy^2(\quad) \)  
(c) \( 90x^4y = 15xy(\quad) \)

(d) \( 24x^6 = -3x^2(\quad) \)  
(e) \( -48x^4y^{10} = -16x^2y^2(\quad) \)  
(f) \( 49x^8y^6 = 7x^4y^3(\quad) \)
4. Use the distributive property to write each of the following products as polynomials.

(a) $4x(5x + 2)$
(b) $-5x(10 - x)$
(c) $6x(x^2 - 4x + 8)$

(d) $-10x^2(2x^2 + x - 8)$
(e) $7xy^3(2x^2y - 5y^5)$
(f) $8x^2y^2(x^3 - 2x^2y + 5xy^2 - y^3)$

(g) $-7x^3(4x^2 + 2x - 1)$
(h) $-16t(2t^2 - 2t + 3)$
(i) $12xy(x^2 - 2xy + y^2)$

5. Fill in the missing part of each product in order to make the equation into an identity.

(a) $10x^3 - 35x^3 = 5x^3(\underline{\phantom{000}})$
(b) $-8x^3y + 2x^2y^2 - 10xy^3 = -2xy(\underline{\phantom{000}})$

(c) $-18t^2 + 45t^5 = -9t^2(\underline{\phantom{000}})$
(d) $45x^4 - 30x^3 + 15x^2 = 15x^2(\underline{\phantom{000}})$

(e) $x(x + 5) + 6(x + 5) = (x + 5)(\underline{\phantom{000}})$
(f) $x^2(x - 3) - (x - 3) = (x - 3)(\underline{\phantom{000}})$

REASONING

Another very important exponent property occurs when we have a monomial with an exponent that is then raised to yet another power. See if you can come up with a general pattern.

6. Write each of the following out as extended products and then simplify. The first is done as an example.

(a) $(x^3)^3 = x^3 \cdot x^3 \cdot x^3 = x^6$
(b) $(x^3)^2 = \underline{\phantom{000}}$

(c) $(x^5)^4 = \underline{\phantom{000}}$
(d) $(x^4)^3 = \underline{\phantom{000}}$

7. So, what is the pattern? For positive integers $a$ and $b$: $(x^a)^b = \underline{\phantom{000}}$
MULTIPLYING POLYNOMIALS
COMMON CORE ALGEBRA II

Polynomials are expressions that are mainly combinations of terms with both addition and subtraction that can have only constants and positive integer powers. They are truly just an extension of our base-10 number system.

Exercise #1: Given the polynomial \(2x^3 + 5x^2 + 3x + 4\), what is its value when \(x = 10\)? How can you determine this without the use of your calculator? If you cannot, use your calculator to help and then explain why the answer turns out as it does.

We've already reviewed how to multiply polynomials by monomials in the last lesson. In this lesson we will look at multiplying polynomials by themselves. The key here is the distributive property. Let's start by looking at the product of binomials.

Exercise #2: Consider the product of \((3x + 2)\) with \((2x + 5)\).

(a) Find this product using the distributive property twice (or possibly "foiling.")

(b) Represent this product on the area model shown below.

Exercise #3: Find the product of the binomial \((4x + 3)\) with the trinomial \((2x^2 - 5x - 3)\). Represent your product using an area array. Even though the result has an \(x^3\) term, the area array can still help us keep track of the product to make sure we are distributing correctly.
It is critical to understand that when we multiply two polynomials then our result is equivalent to this product and this equivalence can be tested.

**Exercise #4:** Consider the product of \((x - 2)\) and \((2x - 5)\).

(a) Evaluate this product for \(x = 4\). Show the work that leads to your result.

(b) Find a trinomial that represents the product of these two binomials.

(c) Evaluate the trinomial for \(x = 4\). Is it equivalent to the answer you found in (a)?

(d) What is the value of the trinomial when \(x = 2\)? Can you explain why this makes sense based on the two binomials?

**Exercise #5:** The product of three binomials, just like the product of two, can be found with repeated applications of the distributive property.

(a) Find the product: \((x - 2)(x + 4)(x - 7)\). Use area arrays to help keep track of the product.

(b) For what three values of \(x\) will the cubic polynomial that you found in part (a) have a value of zero? What famous law is this known as?

(c) Test one of the three values you found in (b) to verify that it is a zero of the cubic polynomial.
**MULTIPLYING POLYNOMIALS**

**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

1. Multiply the following binomials and express each product as an equivalent trinomial. Use an area model to help find your product, if necessary.

   (a) \((x + 5)(x + 8)\) \hspace{1cm} (b) \((3x + 2)(2x - 7)\) \hspace{1cm} (c) \((5x - 2)(2x - 3)\)

   (d) \((x^2 - 4)(x^2 + 10)\) \hspace{1cm} (e) \((2x^3 + 1)(5x^3 + 4)\) \hspace{1cm} (f) \((x^2 - 1)(x^2 - 9)\)

2. Find each of the following products in equivalent form. Use an array model to help find your final answers if you find it helpful.

   (a) \((x + 5)(x^2 + 3x + 2)\) \hspace{1cm} (b) \((2x - 3)(4x^2 + 5x - 7)\)

   (c) \((2x + 5)^3\)
APPLICATIONS

3. A square of an unknown side length $x$ inches has one side length increased by 4 inches and the other increased by 7 inches.

(a) If the original square is shown below with side lengths marked as $x$, label the second diagram to represent the new rectangle constructed by increasing the sides as described above.

(b) Label each portion of the second diagram with their areas in terms of $x$ (when applicable). State the product of $(x + 4)$ and $(x + 7)$ as a trinomial below.

(c) If the original square had a side length of $x = 2$ inches, then what is the area of the second rectangle? Show how you arrived at your answer.

(d) Verify that the trinomial you found in part (b) has the same value as (c) for $x = 2$.

REASONING

4. Think about the expression $(x - 8)(x + 4)$.

(a) For what values of $x$ will this expression be equal to zero? Show how you arrived at your answer.

(b) Write this product as an equivalent trinomial.

(c) Show that this trinomial is also equal to zero at the larger value of $x$ from part (a).
USING TABLES ON YOUR CALCULATOR
COMMON CORE ALGEBRA II

The graphing calculator is an amazing device that can do many things. One function that it is particularly good at is evaluating expressions for different input values. We will be looking at two tools on the calculator today, the STORE feature and TABLES. First let's look at how to use STORE.

Exercise #1: Find the value of each of the following expressions by using the STORE feature on your calculator.

(a) \(x^2 - 2x + 7\) for \(x = 5\)  
(b) \(\frac{2x + 6}{3x - 5}\) for \(x = -10\)  
(c) \(|7x - 20| + x^2\) for \(x = 2\)

Sometimes the calculator can even tell us useful information even when it has a hard time evaluating an expression.

Exercise #2: Consider the expression \(\sqrt{6 - 2x}\). What happens when you try to use STORE to evaluate this expression for \(x = 5\)? Evaluate the expression by hand to help explain what the calculator is trying to tell us.

Exercise #3: Let's work with the product of two binomials again, specifically \((3x + 2)\) and \((x + 5)\).

(a) Find their product in trinomial form.  
(b) Evaluate both the trinomial and the original product for \(x = -2\). What do you notice?

(c) Use the STORE command to evaluate the trinomial from (a) for \(x = -5\). Why does the value of the trinomial turn out to be this specific value at \(x = -5\)? Explain.
The **STORE** feature is extremely helpful when you are trying to determine the value of an expression at one or two input values of \( x \). But, if you want to know an expression's value for multiple inputs, then **TABLES** are a much better tool.

**Exercise #4:** The expression \( x^3 + 2x^2 - 16x - 32 \) has an **integer zero** somewhere on the interval \( 0 \leq x \leq 10 \). Use a **TABLE** to find the **zero** on this interval. Show the table.

Table commands can be particularly good at establishing proof that two expressions are equivalent. This is particularly helpful when you've done a number of manipulations and you want to have confidence that you've produced an algebraically equivalent expression.

**Exercise #5:** Consider the more complex algebraic expression shown below:

\[
(x + 5)(x + 8) - (x + 3)(x - 2)
\]

(a) This relatively complex expression simplifies into a linear binomial expression. Determine this expression carefully. Show your work below.

(b) Set up a table using the original expression and the one you found in (a) over the interval \( 0 \leq x \leq 5 \). Compare values to determine if you correctly simplified the original expression.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 = )</th>
<th>( y_2 = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
USING TABLES ON YOUR CALCULATOR
COMMON CORE ALGEBRA II HOMEWORK

**FLUENCY**

1. Use the **STORE** feature on your calculator to evaluate each of the following. No work needs to be shown.

   (a) \(7x + 18\) for \(x = -8\)
   (b) \(3x^2 - 2x + 5\) for \(x = 3\)
   (c) \(x^3 + 5x^2 - 4x - 20\) for \(x = -5\)

   (d) \(|x^2 - 2x - 8|\) for \(x = 1\)
   (e) \(\frac{5x - 3}{4x^2 + 5}\) for \(x = 2\)
   (f) \(\sqrt{\frac{4-x}{x+9}}\) for \(x = -5\)

2. The **STORE** features is particularly helpful in checking to see if a value is a solution to an equation. Let's see how this works in this problem. Consider the relatively easy linear equation:

   \[6x - 3 = 4x + 9\]

   (a) Solve this equation for \(x\).

   (b) Using **STORE**, determine the value of both the left hand expression, \(6x - 3\), and the right hand expression, \(4x + 9\), at the value of \(x\) you found in (a).

   (c) Why does what you found in part (b) verify that your solution is correct (or possibly incorrect if you made a mistake in (a))?

3. Two of the following values of \(x\) are solutions to the equation: \(x^2 + 4x - 12 = 10x + 4\). Determine which they are and provide a justification for your answer.

   \(x = -2\) \hspace{1cm} \(x = -5\) \hspace{1cm} \(x = 6\) \hspace{1cm} \(x = 8\)
4. The quadratic expression $x^2 - 8x + 10$ has its smallest value for some integer value of $x$ on the interval $0 \leq x \leq 10$. Set up a **TABLE** to find the smallest value of the expression and the value of $x$ that gives this value. Show your table below.

5. Consider the complex expression $(x + 7)(x + 3) + (x - 1)(x - 4)$.

   (a) Multiply the two sets of binomials and combine like terms in order to write this expression as an equivalent trinomial in standard form. Show your work.

   (b) Set up a **TABLE** to verify that your answer in part (a) is equivalent to the original expression. Don't hesitate to point out that it is not equivalent (which means you either made a mistake in your algebra or in your table set up). Show your table.

6. The product of three binomials is shown below. Write this product as a polynomial in standard form. (Its highest power will be $x^3$).

   $$ (x - 1)(x + 2)(x - 4) $$

7. Set up a table for the answer you found in #6 on the interval $-5 \leq x \leq 5$. Where does this expression have **zeroes**?
UNIT #2

FUNCTIONS AS THE CORNERSTONES OF ALGEBRA

Lesson #1 – Introduction to Functions
Lesson #2 – Function Notation
Lesson #3 – Function Composition
Lesson #4 – The Domain and Range of a Function
Lesson #5 – One to One Functions
Lesson #6 – Inverse Functions
Lesson #7 – Key Features of Functions
**INTRODUCTION TO FUNCTIONS**

**COMMON CORE ALGEBRA II**

Most higher level mathematics is built upon the concept of a function. Like most of the important concepts in mathematics, the definition of a function is simple to the point of being easily overlooked. Make sure to know the following definition:

**DEFINITION:** A function is any “rule” that assigns exactly one output value (y-value) for each input value (x-value). These rules can be expressed in different ways, the most common being equations, graphs, and tables of values. We call the input variable **independent** and output variable **dependent**.

**Exercise #1:** An internet music service offers a plan whereby users pay a flat monthly fee of $5 and can then download songs for 10 cents each.

(a) What are the independent and dependent variables in this scenario?

Independent: __________  
Dependent: __________

(b) Fill in the table below for a variety of independent values:

<table>
<thead>
<tr>
<th>Number of downloads, x</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount Charged, y</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Let the number of downloads be represented by the variable \( x \) and the amount charged in dollars be represented by the variable \( y \), write an equation that models \( y \) as a function of \( x \).

(d) Based on the equation you found in part (c), produce a graph of this function for all values of \( x \) on the interval \( 0 \leq x \leq 40 \). Use a calculator **TABLE** to generate additional coordinate pairs to the ones you found in part (b).
Exercise #2: One of the following graphs shows a relationship where $y$ is a function of $x$ and one does not.

(a) Draw the vertical line whose equation is $x = 3$ on both graphs.

(b) Give all output values for each graph at an input of 3.
   
   Relationship A: 
   Relationship B:

(c) Explain which of these relationships is a function and why.

Exercise #3: The graph of the function $y = x^2 - 4x + 1$ is shown below.

(a) State this function’s $y$-intercept.

(b) Between what two consecutive integers does the larger $x$-intercept lie?

(c) Draw the horizontal line $y = -2$ on this graph.

(d) Using these two graphs, find all values of $x$ that solve the equation below:

$$x^2 - 4x + 1 = -2$$

(e) Verify that these values of $x$ are solutions by using STORE on your graphing calculator.
INTRODUCTION TO FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Determine for each of the following graphed relationships whether \( y \) is a function of \( x \) using the Vertical Line Test.

(a)  
(b)  
(c)  
(d)  
(e)  
(f) 

2. What are the outputs for an input of \( x = 5 \) given functions defined by the following formulas:

(a) \( y = 3x - 4 \)  
(b) \( y = 50 - 2x^2 \)  
(c) \( y = 2^x \)
APPLICATIONS

3. Evin is walking home from the museum. She starts 38 blocks from home and walks 2 blocks each minute. Evin’s distance from home is a function of the number of minutes she has been walking.

   (a) Which variable is independent and which variable is dependent in this scenario?

(b) Fill in the table below for a variety of time values.

<table>
<thead>
<tr>
<th>Time, ( t ), in minutes</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from home, ( D ), in blocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Determine an equation relating the distance, \( D \), that Evin is from home as a function of the number of minutes, \( t \), that she has been walking.

(d) Determine the number of minutes, \( t \), that it takes for Evin to reach home.

REASONING

4. In one of the following tables, the variable \( y \) is a function of the variable \( x \). Explain which relationship is a function and why the other is not.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>11</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
<td>4</td>
<td>-2</td>
</tr>
<tr>
<td>6</td>
<td>43</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Relationship #1

Relationship #2
FUNCTION NOTATION

COMMON CORE ALGEBRA II

Functions are fundamental tools that convert inputs, values of the independent variable, to outputs, values of the dependent variable. There is a special notation that is commonly used to show this conversion process. The first exercise will illustrate this notation in the context of formulas.

Exercise #1: Evaluate each of the following given the function definitions and input values.

(a) \( f(x) = 5x - 2 \)  
\( f(3) = \)  
\( f(-2) = \)

(b) \( g(x) = x^2 + 4 \)  
\( g(3) = \)  
\( g(0) = \)

(c) \( h(x) = 2^x \)  
\( h(3) = \)  
\( h(-2) = \)

Although this notation could be confused with multiplication, the context will make it clear that it is not. The idea of function notation is summarized below.

<table>
<thead>
<tr>
<th>Output</th>
<th>Rule</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = f(x) )</td>
<td>( f )</td>
<td>( x )</td>
</tr>
</tbody>
</table>

Recall that function rules commonly come in one of three forms: (1) equations (as in Exercise #1), (2) graphs, and (3) tables. The next few exercises will illustrate function notation with these three forms.

Exercise #2: Boiling water at 212 degrees Fahrenheit is left in a room that is at 65 degrees Fahrenheit and begins to cool. Temperature readings are taken each hour and are given in the table below. In this scenario, the temperature, \( T \), is a function of the number of hours, \( h \).

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
h \text{ (hours)} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
T(h) \text{ (°F)} & 212 & 141 & 104 & 85 & 76 & 70 & 68 & 66 & 65 \\
\hline
\end{array}
\]

(a) Evaluate \( T(2) \) and \( T(6) \).

(b) For what value of \( h \) is \( T(h) = 76 \)?

(c) Between what two consecutive hours will \( T(h) = 100 \)?
**Exercise #3:** The function $y = f(x)$ is defined by the graph shown below. Answer the following questions based on this graph.

(a) Evaluate $f(-1), f(1),$ and $f(5)$.

(b) Evaluate $f(0)$. What special feature on a graph does $f(0)$ always correspond to?

(c) What values of $x$ solve the equation $f(x) = 0$? What special features on a graph does the set of $x$-values that solve $f(x) = 0$ correspond to?

(d) Between what two consecutive integers does the largest solution to $f(x) = 3$ lie?

**Exercise #4:** For a function $y = g(x)$ it is known that $g(-2) = 7$. Which of the following points must lie on the graph of $g(x)$?

(1) $(7, -2)$  
(2) $(-2, 7)$

(3) $(0, 7)$  
(4) $(-2, 0)$

**Exercise #5:** Physics students drop a ball from the top of a 50 foot high building and model its height as a function of time with the equation $h(t) = 50 - 16t^2$. Using **Tables** on your calculator, determine, to the nearest tenth of a second, when the ball hits the ground. Provide tabular outputs to support your answer.
FUNCTION NOTATION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Without using your calculator, evaluate each of the following given the function definitions and input values.

   (a) \( f(x) = 3x + 7 \)
   \[ f(-4) = \]
   \[ f(2) = \]

   (b) \( g(x) = 3x^2 \)
   \[ g(2) = \]
   \[ g(-3) = \]

   (c) \( h(x) = \sqrt{x - 5} \)
   \[ h(41) = \]
   \[ h(14) = \]

2. Using \textit{STORE} on your calculator, evaluate each of the following more complex functions.

   (a) \( f(x) = \frac{3x^2 - 5}{4x + 10} \)
   \[ f(-5) = \]
   \[ f(0) = \]

   (b) \( g(x) = \frac{\sqrt{25 - x^2}}{x} \)
   \[ g(4) = \]
   \[ g(-3) = \]

   (c) \( h(x) = 30(1.2)^x \)
   \[ h(3) = \]
   \[ h(0) = \]

3. Based on the graph of the function \( y = g(x) \) shown below, answer the following questions.

   (a) Evaluate \( g(-2), g(0), g(3) \) and \( g(7) \).

   (b) What values of \( x \) solve the equation \( g(x) = 0 \)?

   (c) Graph the horizontal line \( y = 2 \) on the grid above and label.

   (d) How many values of \( x \) solve the equation \( g(x) = 2 \)?
APPLICATIONS

4. Ian invested $2500 in an investment vehicle that is guaranteed to earn 4% interest compounded yearly. The amount of money, $A$, in his account as a function of the number of years, $t$, since creating the account is given by the equation $A(t) = 2500(1.04)^t$.

(a) Evaluate $A(0)$ and $A(10)$.

(b) What do the two values that you found in part (a) represent?

(c) Using tables on your calculator, determine, to the nearest whole year, the value of $t$ that solves the equation $A(t) = 5000$. Justify your answer with numerical evidence.

(d) What does the value of $t$ that you found in part (b) represent about Ian’s investment?

5. A ball is shot from an air-cannon at an angle of 45° with the horizon. It travels along a path given by the equation $h(d) = -\frac{1}{50}d^2 + d$, where $h$ represents the ball’s height above the ground and $d$ represents the distance the ball has traveled horizontally. Using your calculator to generate a table of values, graph this function for all values of $d$ on the interval $0 \leq d \leq 50$. Look at the table to properly scale the $y$-axis.

What is the maximum height that the ball reaches? At what value of $d$ does it reach this height?
FUNCTION COMPOSITION
COMMON CORE ALGEBRA II

Since functions convert the value of an input variable into the value of an output variable, it stands to reason that this output could then be used as an input to a second function. This process is known as composition of functions, in other words, combining the action or rules of two functions.

**Exercise #1:** A circular garden with a radius of 15 feet is to be covered with topsoil at a cost of $1.25 per square foot of garden space.

(a) Determine the area of this garden to the nearest square foot.  
(b) Using your answer from (a), calculate the cost of covering the garden with topsoil.

In this exercise, we see that the output of an area function is used as the input to a cost function. This idea can be generalized to generic functions, \(f\) and \(g\) as shown in the diagram below.

![Diagram of function composition](image)

There are two notations that are used to indicate composition of two functions. These will be introduced in the next few exercises, both with equations and graphs.

**Exercise #2:** Given \(f(x) = x^2 - 5\) and \(g(x) = 2x + 3\) find values for each of the following.

(a) \(f(g(1))\) =  
(b) \(g(f(2))\) =  
(c) \(g(g(0))\) =  
(d) \((f \circ g)(-2)\) =  
(e) \((g \circ f)(3)\) =  
(f) \((f \circ f)(-1)\) =
Exercise #3: The graphs below are of the functions $y = f(x)$ and $y = g(x)$. Evaluate each of the following questions based on these two graphs.

(a) $g(f(2)) =$ 
(b) $f(g(-1)) =$ 
(c) $g(g(1)) =$ 
(d) $(g \circ f)(-2) =$ 
(e) $(f \circ g)(0) =$ 
(f) $(f \circ f)(0) =$

On occasion, it is desirable to create a formula for the composition of two functions. We will see this facet of composition throughout the course as we study functions. The next two exercises illustrate the process of finding these equations with simple linear and quadratic functions.

Exercise #4: Given the functions $f(x) = 3x - 2$ and $g(x) = 5x + 4$, determine formulas in simplest $y = ax + b$ form for:

(a) $f(g(x))$  
(b) $g(f(x))$

Exercise #5: If $f(x) = x^2$ and $g(x) = x - 5$ then $f(g(x)) =$

(1) $x^2 + 25$  
(2) $x^2 - 25$  
(3) $x^3 - 5$  
(4) $x^3 - 10x + 25$
FUNCTION COMPOSITION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Given \( f(x) = 3x - 4 \) and \( g(x) = -2x + 7 \) evaluate:

   (a) \( f(g(0)) \)  
   (b) \( g(f(-2)) \)  
   (c) \( f(f(3)) \)

   (d) \((g \circ f)(6)\)  
   (e) \((f \circ g)(5)\)  
   (f) \((g \circ g)(2)\)  

2. Given \( h(x) = x^2 + 11 \) and \( g(x) = \sqrt{x-2} \) evaluate:

   (a) \( h(g(18)) \)  
   (b) \( g(h(4)) \)  
   (c) \((g \circ g)(11)\)  

   (d) \( h(h(0)) \)  
   (e) \((h \circ g)(38)\)  
   (f) \((g \circ h)(0)\)  

3. The graphs of \( y = h(x) \) and \( y = k(x) \) are shown below. Evaluate the following based on these two graphs.

   (a) \( h(k(-2)) \)  
   (b) \((k \circ h)(0)\)  
   (c) \( h(h(-2)) \)  
   (d) \((k \circ k)(-2)\)
4. If \( g(x) = 3x - 5 \) and \( h(x) = 2x - 4 \) then \((g \circ h)(x) = \) ?

(a) \(6x - 17\)  
(b) \(3x - 9\)  
(c) \(5x - 9\)  
(d) \(x - 1\)

5. If \( f(x) = x^2 + 5 \) and \( g(x) = x + 4 \) then \( f(g(x)) = \)

(a) \(x^2 + 9\)  
(b) \(4x^2 + 20\)  
(c) \(x^2 + 8x + 21\)  
(d) \(x^2 + 21\)

APPLICATIONS

6. Scientists modeled the intensity of the sun, \( I \), as a function of the number of hours since 6:00 a.m., \( h \), using the function \( I(h) = \frac{12h - h^2}{36} \). They then model the temperature of the soil, \( T \), as a function of the intensity using the function \( T(I) = \sqrt{5000I} \). Which of the following is closest to the temperature of the soil at 2:00 p.m.?

(a) 54  
(b) 84  
(c) 67  
(d) 38

7. Physics students are studying the effect of the temperature, \( T \), on the speed of sound, \( S \). They find that the speed of sound in meters per second is a function of the temperature in degrees Kelvin, \( K \), by \( S(K) = \sqrt{410K} \). The degrees Kelvin is a function of the temperature in Celsius given by \( K(C) = C + 273.15 \). Find the speed of sound when the temperature is 30 degrees Celsius. Round to the nearest tenth.

REASONING

8. Consider the functions \( f(x) = 2x + 9 \) and \( g(x) = \frac{x - 9}{2} \). Calculate the following.

(a) \( g(f(15)) \)  
(b) \( g(f(-3)) \)  
(c) \( g(f(x)) \)

(d) What appears to always be true when you compose these two functions?
THE DOMAIN AND RANGE OF A FUNCTION
COMMON CORE ALGEBRA II

Because functions convert values of inputs into value of outputs, it is natural to talk about the sets that represent these inputs and outputs. The set of inputs that result in an output is called the domain of the function. The set of outputs is called the range.

Exercise #1: Consider the function that has as its inputs the months of the year and as its outputs the number of days in each month. In this case, the number of days is a function of the month of the year. Assume this function is restricted to non-leap years.

(a) Write, in roster form, the set that represents this function’s domain.
(b) Write, in roster form, the set that represents this function’s range.

Exercise #2: State the range of the function \( f(n) = 2n + 1 \) if its domain is the set \( \{1, 3, 5\} \). Show the domain and range in the mapping diagram below.

Exercise #3: The function \( y = g(x) \) is completely defined by the graph shown below. Answer the following questions based on this graph.

(a) Determine the minimum and maximum \( x \)-values represented on this graph.

(b) Determine the minimum and maximum \( y \)-values represented on this graph.

(c) State the domain and range of this function using set builder notation.
Some functions, defined with graphs or equations, have domains and ranges that stretch out to infinity. Consider the following exercise in which a standard parabola is graphed.

**Exercise #4:** The function \( f(x) = x^2 - 2x - 3 \) is graphed on the grid below. Which of the following represent its domain and range written in interval notation?

- (1) Domain: \([-2, 4]\) Range: \([-4, 6]\)
- (2) Domain: \([-2, 4]\) Range: \((-4, \infty)\)
- (3) Domain: \((-\infty, \infty)\) Range: \([-4, 6]\)
- (4) Domain: \((-2, 4)\) Range: \((-4, 6)\)

For most functions defined by an algebraic formula, the domain consists of the set of all real numbers, given the concise symbol \(\mathbb{R}\). Sometimes, though, there are restrictions placed on the domain of a function by the structure of its formula. Two basic restrictions will be illustrated in the next few exercises.

**Exercise #5:** The function \( f(x) = \frac{2x + 1}{x - 4} \) has outputs given by the following calculator table.

(a) Evaluate \( f(1) \) and \( f(6) \) from the table.

<table>
<thead>
<tr>
<th>x</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-2.5</td>
</tr>
<tr>
<td>3</td>
<td>-7</td>
</tr>
<tr>
<td>4</td>
<td>Error</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>6.5</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

(b) Why does the calculator give an ERROR at \( x = 4 \)?

(c) Are there any values except \( x = 4 \) that are not in the domain of \( f \)? Explain.

**Exercise #6:** Which of the following values of \( x \) would not be in the domain of the function \( y = \sqrt{x + 4} \)? Explain your answer.

- (1) \( x = 0 \)
- (2) \( x = 5 \)
- (3) \( x = -3 \)
- (4) \( x = -8 \)
THE DOMAIN AND RANGE OF A FUNCTION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. A function is given by the set of ordered pairs \( \{(2, 5), (4, 9), (6, 13), (8, 17)\} \). Write its domain and range in roster form.

   Domain: ____________________
   Range: ____________________

2. The function \( h(x) = x^2 + 5 \) maps the domain given by the set \( \{-2, -1, 0, 1, 2\} \). Which of the following sets represents the range of \( h(x) \)?

   (1) \( \{0, 6, 10, 12\} \)
   (2) \( \{5, 6, 7\} \)
   (3) \( \{5, 6, 9\} \)
   (4) \( \{1, 4, 5, 6, 9\} \)

3. Which of the following values of \( x \) would not be in the domain of the function defined by \( f(x) = \frac{x-2}{x+3} \)?

   (1) \( x = -3 \)
   (2) \( x = 2 \)
   (3) \( x = 3 \)
   (4) \( x = -2 \)

4. Determine any values of \( x \) that do not lie in the domain of the function \( f(x) = \frac{3x+2}{2x-10} \). Justify your response.

5. Which of the following values of \( x \) does lie in the domain of the function defined by \( g(x) = \sqrt{2x-7} \)?

   (1) \( x = 0 \)
   (2) \( x = 2 \)
   (3) \( x = 3 \)
   (4) \( x = 5 \)

6. Which of the following would represent the domain of the function \( y = \sqrt{6-2x} \)?

   (1) \( \{x : x > 3\} \)
   (2) \( \{x : x < 3\} \)
   (3) \( \{x : x \leq 3\} \)
   (4) \( \{x : x \geq 3\} \)
7. The function \( y = f(x) \) is completely defined by the graph shown below.

(a) Evaluate \( f(-4), f(3), \) and \( f(6) \).

(b) Draw a rectangle that whose vertices are \((-4, 5), (6, 5), (6, -4), \) and \((-4, -4)\).

(c) State the domain and range of this function using interval notation.

\[
\text{Domain: } \quad \text{Range:}
\]

8. Which of the following represents the range of the quadratic function shown in the graph below?

(1) \((4, \infty)\)  
(2) \((-\infty, 4]\)  
(3) \((-\infty, 4)\)  
(4) \([4, \infty)\)

APPLICATIONS

9. A child starts a piggy bank with $2. Each day, the child receives 25 cents at the end of the day and puts it in the bank. If \( A \) represents the amount of money and \( d \) stands for the number of days then \( A(d) = 2 + 0.25d \) gives the amount of money in the bank as a function of days (think about this formula).

(a) Evaluate \( A(1), A(7), \) and \( A(30) \).

(b) For what value of \( d \) will \( A(d) = $10.50 \).

(c) Explain why the domain does not contain the value \( d = 2.5 \).

(d) Explain why the range does not include the value \( A = $3.10 \).
**ONE-TO-ONE FUNCTIONS**

**COMMON CORE ALGEBRA II**

Functions as rules can be divided into various categories based on shared characteristics. One category is comprised of functions known as one-to-one. The following exercise will be illustrate the difference between a function that is one-to-one and one that is not.

**Exercise #1:** Consider the two simple functions given by the equations $f(x) = 2x$ and $g(x) = x^2$.

(a) Map the domain $\{-2, 0, 2\}$ using each function. Fill in the range and show the mapping arrows.

<table>
<thead>
<tr>
<th>Domain of $f$</th>
<th>Range of $f$</th>
<th>Domain of $g$</th>
<th>Range of $g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
<td></td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

(b) What is fundamentally different between these two functions in terms of how the elements of this domain get mapped to the elements of the range?

**ONE-TO-ONE FUNCTIONS**

A function $f(x)$ is called one-to-one if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.

(In other words, different inputs give different outputs.)

**Exercise #2:** Of the four tables below, one represents a relationship where $y$ is a one-to-one function of $x$. Determine which it is and explain why the others are not.

(1) $\begin{array}{c|c}
 x & y \\
 4 & 2 \\
 4 & -2 \\
 9 & 3 \\
 9 & -3 \\
\end{array}$

(2) $\begin{array}{c|c}
 x & y \\
 -2 & 1 \\
 -1 & 0 \\
 0 & 1 \\
 1 & 2 \\
\end{array}$

(3) $\begin{array}{c|c}
 x & y \\
 1 & 2 \\
 2 & 4 \\
 3 & 8 \\
 4 & 16 \\
\end{array}$

(4) $\begin{array}{c|c}
 x & y \\
 -3 & 10 \\
 -2 & 9 \\
 -1 & 7 \\
 -2 & 10 \\
\end{array}$
**Exercise #3:** Consider the following four graphs which show a relationship between the variables \( y \) and \( x \).

(a) Circle the two graphs above that are functions. Explain how you know they are functions.

(b) Of the two graphs you circled, which is one-to-one? Explain how you can tell from its graph.

---

**THE HORIZONTAL LINE TEST**

If any given horizontal line passes through the graph of a function at most one time, then that function is one-to-one. This test works because horizontal lines represent constant \( y \)-values; hence, if a horizontal line intersects a graph more than once, an output has been repeated.

**Exercise #4:** Which of the following represents the graph of a one-to-one function?

(a) Circle the two graphs above that are functions. Explain how you know they are functions.

(b) Of the two graphs you circled, which is one-to-one? Explain how you can tell from its graph.

**Exercise #5:** The distance that a number, \( x \), lies from the number 5 on a one-dimensional number line is given by the function \( D(x) = |x - 5| \). Show by example that \( D(x) \) is not a one-to-one function.
ONE-TO-ONE FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following graphs illustrates a one-to-one relationship?

(1)  
(2)  
(3)  
(4)  

2. Which of the following graphs does not represent that of a one-to-one function?

(1)  
(2)  
(3)  
(4)  

3. In which of the following graphs is each input not paired with a unique output?

(1)  
(2)  
(3)  
(4)  

4. In which of the following formulas is the variable y a one-to-one function of the variable x? (Hint – try generating some values either in your head or using TABLES on your calculator.)

(1)  \( y = x^2 \)  
(2)  \( y = |x| \)  
(3)  \( y = 2x \)  
(4)  \( y = 5 \)
5. Which of the following tables illustrates a relationship in which \( y \) is a one-to-one function of \( x \)?

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>-8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>-5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### APPLICATIONS

6. A recent newspaper gave temperature data for various days of the week in table format. In which of the tables below is the reported temperature a one-to-one function of the day of the week?

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tue</td>
<td>68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wed</td>
<td>65</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thu</td>
<td>74</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tue</td>
<td>72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wed</td>
<td>68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thu</td>
<td>72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tue</td>
<td>52</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wed</td>
<td>81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thu</td>
<td>76</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>( x )</td>
<td>( y )</td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tue</td>
<td>58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tue</td>
<td>85</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Physics students drop a basketball from 5 feet above the ground and its height is measured each tenth of a second until it stops bouncing. The height of the basketball, \( h \), is clearly a function of the time, \( t \), since it was dropped.

(a) Sketch the general graph of what you believe this function would look like.

(b) Is the height of the ball a one-to-one function of time? Explain your answer.

![Graph](image)

### REASONING

8. Consider the function \( f(x) = \text{round}(x) \), which rounds the input, \( x \), to the nearest integer. Is this function one-to-one? Explain or justify your answer.
The idea of inverses, or opposites, is very important in mathematics. So important, in fact, that the word is used in many different contexts, including the additive and multiplicative inverses of a number. The actions of certain functions can be reversed as well. The rules governing the reversal themselves can be functions.

**Exercise #1:** Consider the two linear functions given by the formulas $f(x) = \frac{3x+7}{2}$ and $g(x) = \frac{2x-7}{3}$.

(a) Calculate $f(5)$ and $g(11)$.  
(b) Calculate $f(0)$ and $g\left(\frac{7}{2}\right)$.  
(c) Calculate $f\left(g(-1)\right)$.

(d) Calculate $f\left(g(5)\right)$.  
(e) Without calculation, determine the value of $f\left(g(\pi)\right)$.

The two functions seen in Exercise #1 are inverses because they literally “undo” one another. The general idea of inverses, $f(x)$ and $g(x)$, is shown below in the mapping diagram.

![Mapping Diagram](image)

**Exercise #2:** If the point $(-3,5)$ lies on the graph of $y = f(x)$, then which of the following points must lie on the graph of its inverse?

(1) $(3,-5)$  
(2) $(-5,3)$  
(3) $(5,-3)$  
(4) $\left(-\frac{1}{3}, \frac{1}{5}\right)$
Inverse functions have their own special notation. It is shown in the box below.

**Inverse Function Notation**

If a function \( y = f(x) \) has an inverse that is also a function we represent it as \( y = f^{-1}(x) \).

**Exercise #3:** The linear function \( f(x) = \frac{2}{3}x - 2 \) is shown graphed below. Use its graph to answer the following questions.

(a) Evaluate \( f^{-1}(2) \) and \( f^{-1}(-4) \).

(b) Determine the \( y \)-intercept of \( f^{-1}(x) \).

(c) On the same set of axes, draw a graph of \( y = f^{-1}(x) \).

**Exercise #4:** A table of values for the simple quadratic function \( f(x) = x^2 \) is given below along with its graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) Graph the inverse by switching the ordered pairs.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f^{-1}(x) )</th>
</tr>
</thead>
</table>

(b) What do you notice about the graph of this function’s inverse?

**Existence of Inverse Functions**

A function will have an inverse that is also a function if and only if it is one-to-one. Hence, a quick way to know if a function has an inverse that is also a function is to apply the Horizontal Line Test.
INVERSE FUNCTIONS

COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. If the point \((-7, 5)\) lies on the graph of \(y = f(x)\), which of the following points must lie on the graph of its inverse?
   
   (1) \((5, -7)\)  
   (2) \(\left(-\frac{1}{7}, \frac{1}{5}\right)\)  
   (3) \((7, -5)\)  
   (4) \(\left(\frac{1}{7}, -\frac{1}{5}\right)\)

2. The function \(y = f(x)\) has an inverse function \(y = f^{-1}(x)\). If \(f(a) = -b\) then which of the following must be true?
   
   (1) \(f^{-1}(-b) = -a\)  
   (2) \(f^{-1}\left(\frac{1}{a}\right) = -\frac{1}{b}\)  
   (3) \(f^{-1}(-b) = a\)  
   (4) \(f^{-1}(b) = -a\)

3. The graph of the function \(y = g(x)\) is shown below. The value of \(g^{-1}(2)\) is
   
   (1) \(2.5\)  
   (2) \(-4\)  
   (3) \(0.4\)  
   (4) \(-1\)

4. Which of the following functions would have an inverse that is also a function?

   (1) \[
   \begin{array}{c}
   \end{array}
   \]
   (2) \[
   \begin{array}{c}
   \end{array}
   \]
   (3) \[
   \begin{array}{c}
   \end{array}
   \]
   (4) \[
   \begin{array}{c}
   \end{array}
   \]

5. For a one-to-one function it is known that \(f(0) = 6\) and \(f(8) = 0\). Which of the following must be true about the graph of this function’s inverse?
   
   (1) its \(y\)-intercept = 6  
   (2) its \(y\)-intercept = 8  
   (3) its \(x\)-intercept = \(-6\)  
   (4) its \(x\)-intercept = \(-8\)
6. The function \( y = h(x) \) is entirely defined by the graph shown below.

(a) Sketch a graph of \( y = h^{-1}(x) \). Create a table of values if needed.

(b) Write the domain and range of \( y = h(x) \) and \( y = h^{-1}(x) \) using interval notation.

\[
\begin{align*}
\text{Domain:} & & \text{Range:} \\
\text{Domain:} & & \text{Range:}
\end{align*}
\]

APPLICATIONS

7. The function \( y = A(r) = \pi r^2 \) is a one-to-one function that uses a circle’s radius as an input and gives the circle’s area as its output. Selected values of this function are shown in the table below.

<table>
<thead>
<tr>
<th>( r )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(r) )</td>
<td>( \pi )</td>
<td>( 4\pi )</td>
<td>( 9\pi )</td>
<td>( 16\pi )</td>
<td>( 25\pi )</td>
<td>( 36\pi )</td>
</tr>
</tbody>
</table>

(a) Determine the values of \( A^{-1}(9\pi) \) and \( A^{-1}(36\pi) \) from using the table.

(b) Determine the values of \( A^{-1}(100\pi) \) and \( A^{-1}(225\pi) \).

(c) The original function \( y = A(r) \) converted an input, the circle’s radius, to an output, the circle’s area. What are the inputs and outputs of the inverse function?

Input: | Output:

REASONING

8. The domain and range of a one-to-one function, \( y = f(x) \), are given below in set-builder notation. Give the domain and range of this function’s inverse also in set-builder notation.

\[
\begin{align*}
\text{Domain:} & & \text{Range:} \\
\text{Domain:} & & \text{Range:}
\end{align*}
\]
KEY FEATURES OF FUNCTIONS
COMMON CORE ALGEBRA II

The graphs of functions have many key features whose terminology we will be using all year. It is important to master this terminology, most of which you learned in Common Core Algebra I.

Exercise #1: The function \( y = f(x) \) is shown graphed to the right.

Answer the following questions based on this graph.

(a) State the \( y \)-intercept of the function.

(b) State the \( x \)-intercepts of the function. What is the alternative name that we give the \( x \)-intercepts?

(c) Over the interval \(-1 < x < 2\) is \( f(x) \) increasing or decreasing? How can you tell?

(d) Give the interval over which \( f(x) > 0 \). What is a quick way of seeing this visually?

(e) State all the \( x \)-coordinates of the relative maximums and relative minimums. Label each.

(f) What are the absolute maximum and minimum values of the function? Where do they occur?

(g) State the domain and range of \( f(x) \) using interval notation.

(h) If a second function \( g(x) \) is defined by the formula \( g(x) = \frac{1}{2} f(x + 2) \), then what is the \( y \)-intercept of \( g \)?
**Exercise #2:** Consider the function $g(x) = 2|x - 1| - 8$ defined over the domain $-4 \leq x \leq 7$.

(a) Sketch a graph of the function to the right.

(b) State the domain interval over which this function is decreasing.

(c) State zeroes of the function on this interval.

(d) State the interval over which $g(x) \leq 0$

(e) Evaluate $g(0)$ by using the algebraic definition of the function. What point does this correspond to on the graph?

You need to be able to think about functions in all of their forms, including equations, graphs, and tables. Tables can be quick to use, but sometimes hard to understand.

**Exercise #3:** A continuous function $f(x)$ has a domain of $-6 \leq x \leq 13$ with selected values shown below. The function has exactly two zeroes and has exactly two turning points, one at $(3, -4)$ and one at $(9, 3)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>-1</th>
<th>0</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>9</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>5</td>
<td>0</td>
<td>-2</td>
<td>-4</td>
<td>-1</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) State the interval over which $f(x) < 0$.

(b) State the interval over which $f(x)$ is increasing.
1. The piecewise linear function \( f(x) \) is shown to the right. Answer the following questions based on its graph.

(a) Evaluate each of the following based on the graph:

(i) \( f(4) = \) 

(ii) \( f(-3) = \)

(b) State the zeroes of \( f(x) \).

(c) Over which of the following intervals is \( f(x) \) always increasing?

(1) \(-7 < x < -3\) 
(3) \(-5 < x < 5\)
(2) \(-3 < x < 5\) 
(4) \(-5 < x < 3\)

(d) State the coordinates of the relative maximum and the relative minimum of this function.

Relative Maximum: ________________
Relative Minimum: ________________

(e) Over which of the following intervals is \( f(x) < 0 \) ?

(1) \(-7 < x < -3\) 
(3) \(-5 < x < 2\)
(2) \(2 \leq x \leq 7\) 
(4) \(-5 \leq x \leq 2\)

(f) A second function \( g(x) \) is defined using the rule \( g(x) = 2f(x) + 5 \). Evaluate \( g(0) \) using this rule. What does this correspond to on the graph of \( g \)?

(g) A third function \( h(x) \) is defined by the formula \( h(x) = x^3 - 3 \). What is the value of \( g(h(2)) \)? Show how you arrived at your answer.
2. For the function \( g(x) = 9 - (x + 1)^2 \) do the following.

(a) Sketch the graph of \( g \) on the axes provided.

(b) State the zeroes of \( g \).

(c) Over what interval is \( g(x) \) decreasing?

(d) Over what interval is \( g(x) \geq 0 ? \)

(e) State the range of \( g \).

3. Draw a graph of \( y = f(x) \) that matches the following characteristics.

Increasing on: \(-8 < x < -4 \) and \(-1 < x < 5 \)

Decreasing on: \(-4 < x < -1 \)

\( f(-8) = -5 \) and zeroes at \( x = -6, -2, \) and 3

Absolute maximum of 7 and absolute minimum of \(-5 \)

4. A continuous function has a domain of \(-7 \leq x \leq 10 \) and has selected values shown in the table below. The function has exactly two zeroes and a relative maximum at \((-4, 12) \) and a relative minimum at \((5, -6) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-7</th>
<th>-4</th>
<th>-1</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8</td>
<td>12</td>
<td>0</td>
<td>-2</td>
<td>-5</td>
<td>-6</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) State the interval on which \( f(x) \) is decreasing.  
(b) State the interval over which \( f(x) < 0 \).
UNIT #3

LINEAR FUNCTIONS, EQUATIONS, AND THEIR ALGEBRA

Lesson #1 – Direct Variation
Lesson #2 – Average Rate of Change
Lesson #3 – Forms of a Line
Lesson #4 – Linear Modeling
Lesson #5 – Inverses of Linear Functions
Lesson #6 – Piecewise Linear Functions
Lesson #7 - Systems of Linear Equations (Primarily 3 by 3)
We begin our linear unit by looking at the simplest linear relationship that can exist between two variables, namely that of **direct variation**. We say that two variables are **directly related** or **proportional** to one another if the following relationship holds.

**Exercise #1:** In each of the following, \(x\) and \(y\) are directly related. Solve for the missing value.

(a) \(y = 15\) when \(x = 5\)  
\(y = ?\) when \(x = 9\)

(b) \(y = -6\) when \(x = 4\)  
\(y = ?\) when \(x = -10\)

(c) \(y = 12\) when \(x = 16\)  
\(y = ?\) when \(x = 24\)

**Exercise #2:** The distance a person can travel varies directly with the time they have been traveling if going at a constant speed. If Phoenix traveled 78 miles in 1.5 hours while going at a constant speed, how far will he travel in 2 hours at the same speed?

**Exercise #3:** Jenna works a job where her pay varies directly with the number of hours she has worked. In one week, she worked 35 hours and made $274.75. How many hours would she need to work in order to earn $337.55?
We will now examine the graph of a direct relationship and see why it is indeed the simplest of all linear functions.

**Exercise #4:** Two variables, \( x \) and \( y \), vary directly. When \( x = 6 \) then \( y = 4 \). The point is shown plotted below.

(a) Find the \( y \)-values for each of the following \( x \)-values. Plot each point and connect.

\[
\begin{align*}
  x &= 3 \\
  x &= -6
\end{align*}
\]

(b) What is the constant of variation in this problem? What does it represent on this line?

(c) Write the equation of the line you plotted in (a).

Direct relationships often exist between two variables whose values are zero simultaneously.

**Exercise #3:** The miles driven by a car, \( d \), varies directly with the number of gallons, \( g \), of gasoline used. Abagail is able to drive \( d = 336 \) miles on \( g = 8 \) gallons of gasoline in her hybrid vehicle.

(a) Calculate the constant of variation for the relationship \( \frac{d}{g} \). Include proper units in your answer.

(b) Give a linear equation that represents the relationship between \( d \) and \( g \). Express your answer as an equation solved for \( d \).

(c) How far can Abagail drive on \( g = 6 \) gallons of gas?

(d) How many gallons of gas will Abagail need in order to drive 483 miles?
1. In each of the following, the variable pair given are proportional to one another. Find the missing value.

(a) \(b = 8\) when \(a = 16\) \(b = ?\) when \(a = 18\)
(b) \(y = 10\) when \(x = 14\) \(y = ?\) when \(x = 21\)
(c) \(w = -2\) when \(u = 6\) \(w = ?\) when \(u = -15\)

2. In the following exercises, the two variables given vary directly with one another. Solve for the missing value.

(a) \(p = 12\) when \(q = 8\) \(p = ?\) when \(q = 6\)
(b) \(y = 21\) when \(x = 9\) \(y = ?\) when \(x = -6\)
(c) \(z = -5\) when \(w = 2\) \(z = ?\) when \(w = 8\)

3. If \(x\) and \(y\) vary directly and \(y = 16\) when \(x = 12\), then which of the following equations correctly represents the relationship between \(x\) and \(y\)?

(1) \(y = \frac{3}{4}x\)
(2) \(y = \frac{4}{3}x\)
(3) \(xy = 192\)
(4) \(y = \frac{4}{3}x\)
APPLICATIONS

4. The distance Max’s bike moves is directly proportional to how many rotations his bike’s crank shaft has made. If Max’s bike moves 25 feet after two rotations, how many feet will the bike move after 15 rotations?

5. For his workout, the increase in Jacob’s heart rate is directly proportional to the amount of time he has spent working out. If his heartbeat has increased by 8 beats per minute after 20 minutes of working out, how much will his heartbeat have increased after 30 minutes of working out?

6. When a photograph is enlarged or shrunken, its width and length stay proportional to the original width and length. Rojas is enlarging a picture whose original width was 3 inches and whose original length was 5 inches. If its new length is to be 8 inches, what is the exact value of its new width in inches?

7. For a set amount of time, the distance Kirk can run is directly related to his average speed. If Kirk can run 3 miles in while running at 6 miles per hour, how far can he run in the same amount of time if his speed increases to 10 miles per hour?

REASONING

8. Two variables are proportional if they can be written at $y = kx$, where $k$ is some constant. This leads to the fact that when $x = 0$ then $y = 0$ as well. Is the temperature measured in Celsius proportional to the temperature measured in Fahrenheit? Explain.
When we model using functions, we are very often interested in the rate that the output is changing compared to the rate of the input.

**Exercise #1:** The function \( f(x) \) is shown graphed to the right.

(a) Evaluate each of the following based on the graph:

\( f(0) \) \hspace{1cm} f(4) \hspace{1cm} f(7) \hspace{1cm} f(13) 

(b) Find the change in the function, \( \Delta f \), over each of the following domain intervals. Find this both by subtraction and show this on the graph.

\( 0 \leq x \leq 4 \hspace{1cm} 4 \leq x \leq 7 \hspace{1cm} 7 \leq x \leq 13 \)

(c) Why can't you simply compare the changes in \( f \) from part (b) to determine over which interval the function changing the fastest?

(d) Calculate the average rate of change for the function over each of the intervals and determine which interval has the greatest rate.

\( 0 \leq x \leq 4 \hspace{1cm} 4 \leq x \leq 7 \hspace{1cm} 7 \leq x \leq 13 \)

(e) Using a straightedge, draw in the lines whose slopes you found in part (d) by connecting the points shown on the graph. The average rate of change gives a measurement of what property of the line?
The average rate of change is an exceptionally important concept in mathematics because it gives us a way to quantify how fast a function changes on average over a certain domain interval. Although we used its formula in the last exercise, we state it formally here:

**Average Rate of Change**

For a function over the domain interval \( a \leq x \leq b \), the function's average rate of change is calculated by:

\[
\frac{\Delta f}{\Delta x} = \frac{\text{change in the output}}{\text{change in the input}} = \frac{f(b) - f(a)}{b - a}
\]

**Exercise #2:** Consider the two functions \( f(x) = 5x + 7 \) and \( g(x) = 2x^2 + 1 \).

(a) Calculate the average rate of change for both functions over the following intervals. Do your work carefully and show the calculations that lead to your answers.

(i) \(-2 \leq x \leq 3\)

(ii) \(1 \leq x \leq 5\)

(b) The average rate of change for \( f \) was the same for both (i) and (ii) but was not the same for \( g \). Why is that?

**Exercise #3:** The table below represents a linear function. Fill in the missing entries.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>5</th>
<th>11</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-5</td>
<td>1</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>
AVERAGE RATE OF CHANGE
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. For the function \( g(x) \) given in the table below, calculate the average rate of change for each of the following intervals.

\[
\begin{array}{c|cccccc}
  x & -3 & -1 & 4 & 6 & 9 \\
g(x) & 8 & -2 & 13 & 12 & 5 \\
\end{array}
\]

(a) \(-3 \leq x \leq -1\)  
(b) \(-1 \leq x \leq 6\)  
(c) \(-3 \leq x \leq 9\)

(d) Explain how you can tell from the answers in (a) through (c) that this is not a table that represents a linear function.

2. Consider the simple quadratic function \( f(x) = x^2 \). Calculate the average rate of change of this function over the following intervals:

(a) \(0 \leq x \leq 2\)  
(b) \(2 \leq x \leq 4\)  
(c) \(4 \leq x \leq 6\)

(d) Clearly the average rate of change is getting larger at \( x \) gets larger. How is this reflected in the graph of \( f \) shown sketched to the right?
3. Which has a greater average rate of change over the interval \(-2 \leq x \leq 4\), the function \(g(x) = 16x - 3\) or the function \(f(x) = 2x^2\)? Provide justification for your answer.

**APPLICATIONS**

4. An object travels such that its distance, \(d\), away from its starting point is shown as a function of time, \(t\), in seconds, in the graph below.

   (a) What is the average rate of change of \(d\) over the interval \(5 \leq t \leq 7\)? Include proper units in your answer.

   (b) The average rate of change of distance over time (what you found in part (a)) is known as the **average speed** of an object. Is the average speed of this object greater on the interval \(0 \leq t \leq 5\) or \(11 \leq t \leq 14\)? Justify.

**REASONING**

5. What makes the average rate of change of a linear function different from that of any other function? What is the special name that we give to the average rate of change of a linear function?
LINEAR FUNCTIONS, EQUATIONS, AND THEIR ALGEBRA – LESSON #3

FORMS OF A LINE
COMMON CORE ALGEBRA II

Linear functions come in a variety of forms. The two shown below have been introduced in Common Core Algebra I and Common Core Geometry.

**TWO COMMON FORMS OF A LINE**

**Slope-Intercept:** \( y = mx + b \)

**Point-Slope:** \( y - y_1 = m(x - x_1) \)

where \( m \) is the slope (or average rate of change) of the line and \((x_1, y_1)\) represents one point on the line.

**Exercise #1:** Consider the linear function \( f(x) = 3x + 5 \).

(a) Determine the \( y \)-intercept of this function by evaluating \( f(0) \).

(b) Find its average rate of change over the interval \(-2 \leq x \leq 3\).

**Exercise #2:** Consider a line whose slope is 5 and which passes through the point \((-2, 8)\).

(a) Write the equation of this line in point-slope form, \( y - y_1 = m(x - x_1) \).

(b) Write the equation of this line in slope-intercept form, \( y = mx + b \).

**Exercise #3:** Which of the following represents an equation for the line that is parallel to \( y = \frac{3}{2}x - 7 \) and which passes through the point \((6, -8)\)?

(1) \( y - 8 = -\frac{2}{3}(x + 6) \)

(2) \( y - 8 = \frac{3}{2}(x + 6) \)

(3) \( y + 8 = \frac{3}{2}(x - 6) \)

(4) \( y + 8 = -\frac{2}{3}(x - 6) \)
Exercise #4: A line passes through the points \((5, -2)\) and \((20, 4)\).

(a) Determine the slope of this line in simplest rational form. 
(b) Write an equation of this line in point-slope form.

(c) Write an equation for this line in slope-intercept form.
(d) For what \(x\)-value will this line pass through a \(y\)-value of 12?

Exercise #5: The graph of a linear function is shown below.

(a) Write the equation of this line in \(y = mx + b\) form.

(b) What must be the slope of a line perpendicular to the one shown?

(c) Draw a line perpendicular to the one shown that passes through the point \((1, 3)\).

(d) Write the equation of the line you just drew in point-slope form.

(e) Does the line that you drew contain the point \((30, -15)\)? Justify.
FORMS OF A LINE
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following lines is perpendicular to \( y = \frac{5}{3}x - 7 \) and has a \( y \)-intercept of 4?

   (1) \( y = \frac{5}{3}x + 4 \)  
   (2) \( y = -\frac{3}{5}x + 4 \)  
   (3) \( y = 4x - \frac{3}{5} \)  
   (4) \( y = \frac{3}{5}x + 4 \)

2. Which of the following lines passes through the point \((-4, -8)\)?

   (1) \( y + 8 = 3(x + 4) \)  
   (2) \( y - 8 = 3(x - 4) \)  
   (3) \( y + 8 = 3(x - 4) \)  
   (4) \( y - 8 = 3(x + 4) \)

3. Which of the following equations could describe the graph of the linear function shown below?

   (1) \( y = \frac{2}{3}x - 4 \)  
   (2) \( y = \frac{2}{3}x + 4 \)  
   (3) \( y = -\frac{2}{3}x - 4 \)  
   (4) \( y = -\frac{2}{3}x + 4 \)

4. For a line whose slope is \(-3\) and which passes through the point \((5, -2)\):

   (a) Write the equation of this line in point-slope form, \( y - y_1 = m(x - x_1) \).
   (b) Write the equation of this line in slope-intercept form, \( y = mx + b \).

5. For a line whose slope is 0.8 and which passes through the point \((-3, 1)\):

   (a) Write the equation of this line in point-slope form, \( y - y_1 = m(x - x_1) \).
   (b) Write the equation of this line in slope-intercept form, \( y = mx + b \).
**REASONING**

6. The two points \((-3, 6)\) and \((6, 0)\) are plotted on the grid below.

   (a) Find an equation, in \(y = mx + b\) form, for the line passing through these two points. Use of the grid is optional.

   (b) Does the point \((-30, -16)\) lie on this line? Justify.

7. A linear function is graphed below along with the point \((3, 1)\).

   (a) Draw a line parallel to the one shown that passes through the point \((3, 1)\).

   (b) Write an equation for the line you just drew in point-slope form.

   (c) Between what two consecutive integers does the \(y\)-intercept of the line you drew fall?

   (d) Determine the exact value of the \(y\)-intercept of the line you drew.
In Common Core Algebra I, you used linear functions to model any process that had a constant rate at which one variable changes with respect to the other, or a constant slope. In this lesson we will review many of the facets of this type of modeling.

**Exercise #1:** Dia was driving away from New York City at a constant speed of 58 miles per hour. He started 45 miles away.

(a) Write a linear function that gives Dia’s distance, $D$, from New York City as a function of the number of hours, $h$, he has been driving. 

(b) If Dia’s destination is 270 miles away from New York City, algebraically determine to the nearest tenth of an hour how long it will take Dia to reach his destination.

In Exercise #1, it is clear from the context what both the slope and the $y$-intercept of this linear model are. Although this is often the case when constructing a linear model, sometimes the slope and a point are known, in which case, the point slope form of the a line is more appropriate.

**Exercise #2:** Edelyn is trying to model her cell-phone plan. She knows that it has a fixed cost, per month, along with a $0.15 charge per call she makes. In her last month’s bill, she was charged $12.80 for making 52 calls.

(a) Create a linear model, in point-slope form, for the amount Edelyn must pay, $P$, per month given the number of phone calls she makes, $c$.

(b) How much is Edelyn’s fixed cost? In other words, how much would she have to pay for making zero phone calls?
Many times linear models have been constructed and we are asked only to work with these models. Models in the real world can be messy and it is often convenient to use our graphing calculators to plot and investigate their behavior.

**Exercise #3:** A factory produces widgets (generic objects of no particular use). The cost, $C$, in dollars to produce $w$ widgets is given by the equation $C = 0.18w + 20.64$. Each widget sells for 26 cents. Thus, the revenue gained, $R$, from selling these widgets is given by $R = 0.26w$.

(a) Use your graphing calculator to sketch and label each of these linear functions for the interval $0 \leq w \leq 500$. Be sure to label your $y$-axis with its scale.

(b) Use your calculator’s INTERSECT command to determine the number of widgets, $w$, that must be produced for the revenue to equal the cost.

(c) If profit is defined as the revenue minus the cost, create an equation in terms of $w$ for the profit, $P$.

(d) Using your graphing calculator, sketch a graph of the profit over the interval $0 \leq w \leq 1000$. Use a TABLE on your calculator to determine an appropriate WINDOW for viewing. Label the $x$ and $y$ intercepts of this line on the graph.

(e) What is the minimum number of widgets that must be sold in order for the profit to reach at least $40? Illustrate this on your graph.
LINEAR MODELING
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. Which of the following would model the distance, $D$, a driver is from Chicago if they are heading towards the city at 58 miles per hour and started 256 miles away?

   (1) $D = 256t + 58$
   (2) $D = 256 - 58t$
   (3) $D = 58t + 256$
   (4) $D = 58 - 256t$

2. The cost, $C$, of producing $x$-bikes is given by $C = 22x + 132$. The revenue gained from selling $x$-bikes is given by $R = 350x$. If the profit, $P$, is defined as $P = R - C$, then which of the following is an equation for $P$ in terms of $x$?

   (1) $P = 328x - 132$
   (2) $P = 372x + 132$
   (3) $P = 328x + 132$
   (4) $P = 372x - 132$

3. The average temperature of the planet is expected to rise at an average rate of 0.04 degrees Celsius per year due to global warming. The average temperature in the year 2000 was 14.71 degrees Celsius. The average Celsius temperature, $C$, is given by $C = 14.71 + 0.04x$, where $x$ represents the number of years since 2000.

   (a) What will be the average temperature in the year 2100?
   (b) Algebraically determine the number of years, $x$, it will take for the temperature, $C$, to reach 20 degrees Celsius. Round to the nearest year.
   (c) Sketch a graph of the average yearly temperature below for the interval $0 \leq x \leq 200$. Be sure to label your $y$-axis scale as well as two points on the line (the $y$-intercept and one additional point).
   (d) What does this model project to be the average global temperature in 2200?
4. Fabio is driving west away from Albany and towards Buffalo along Interstate 90 at a constant rate of speed of 62 miles per hour. After driving for 1.5 hours, Fabio is 221 miles from Albany.

(a) Write a linear model for the distance, \( D \), that Fabio is away from Albany as a function of the number of hours, \( h \), that he has been driving. Write your model in point-slope form, \( D - D_i = m(h - h_i) \).

(b) Rewrite this model in slope-intercept form, \( D = mh + b \).

(c) How far was Fabio from Albany when he started his trip?

(d) If the total distance from Albany to Buffalo is 290 miles, determine how long it takes for Fabio to reach Buffalo. Round your answer to the nearest tenth of an hour.

5. A particular rocket taking off from the Earth’s surface uses fuel at a constant rate of 12.5 gallons per minute. The rocket initially contains 225 gallons of fuel.

(a) Determine a linear model, in \( y = ax + b \) form, for the amount of fuel, \( y \), as a function of the number of minutes, \( x \), that the rocket has burned.

(b) Below is a general sketch of what the graph of your model should look like. Using your calculator, determine the \( x \) and \( y \) intercepts of this model and label them on the graph at points A and B respectively.

(c) The rocket must still contain 50 gallons of fuel when it hits the stratosphere. What is the maximum number of minutes the rocket can take to hit the stratosphere? Show this point on your graph by also graphing the horizontal line \( y = 50 \) and showing the intersection point.
INVERSES OF LINEAR FUNCTIONS
COMMON CORE ALGEBRA II

Recall that functions have inverses that are also functions if they are one-to-one. With the exception of horizontal lines, all linear functions are one-to-one and thus have inverses that are also functions. In this lesson we will investigate these inverses and how to find their equations.

Exercise #1: On the grid below the linear function \( y = 2x - 4 \) is graphed along with the line \( y = x \).

(a) How can you quickly tell that \( y = 2x - 4 \) is a one-to-one function?

(b) Graph the inverse of \( y = 2x - 4 \) on the same grid. Recall that this is easily done by switching the \( x \) and \( y \) coordinates of the original line.

(c) What can be said about the graphs of \( y = 2x - 4 \) and its inverse with respect to the line \( y = x \)?

(d) Find the equation of the inverse in \( y = mx + b \) form.

(e) Find the equation of the inverse in \( y = \frac{x + b}{a} \) form.

As we can see from part (e) in Exercise #1, inverses of linear functions include the inverse operations of the original function but in reverse order. This gives rise to a simple method of finding the equation of any inverse. Simply switch the \( x \) and \( y \) variables in the original equation and solve for \( y \).

Exercise #2: Which of the following represents the equation of the inverse of \( y = 5x - 20 \)?

(1) \( y = \frac{1}{5}x + 20 \)  
(2) \( y = \frac{1}{5}x - 20 \)  
(3) \( y = \frac{1}{5}x - 4 \)  
(4) \( y = \frac{1}{5}x + 4 \)
Although this is a simple enough procedure, certain problems can lead to common errors when solving for \( y \). Care should be taken with each algebraic step.

**Exercise #3:** Which of the following represents the inverse of the linear function \( y = \frac{2}{3}x + 8 \)?

(1) \( y = \frac{3}{2}x - 8 \)  
(2) \( y = \frac{3}{2}x - 12 \)

(3) \( y = -\frac{3}{2}x + 8 \)  
(4) \( y = -\frac{3}{2}x + 12 \)

**Exercise #4:** What is the \( y \)-intercept of the inverse of \( y = \frac{3}{5}x - 9 \)?

(1) \( y = 15 \)  
(2) \( y = \frac{1}{9} \)

(3) \( y = -9 \)  
(4) \( y = -\frac{5}{3} \)

Sometimes we are asked to work with linear functions in their point-slope form. The method of finding the inverse and plotting it, though, do not change just because the linear equation is written in a different form.

**Exercise #5:** Which of the following would be an equation for the inverse of \( y + 6 = 4(x - 2) \)?

(1) \( y - 2 = \frac{1}{4}(x + 6) \)  
(2) \( y - 2 = -\frac{1}{4}(x + 6) \)

(3) \( y - 6 = -4(x + 2) \)  
(4) \( y + 2 = -4(x - 6) \)

**Exercise #6:** Which of the following points lies on the graph of the inverse of \( y - 8 = 5(x + 2) \)? Explain your choice.

(1) \((8, -2)\)  
(2) \((-8, 2)\)

(3) \((-10, 40)\)  
(4) \((-2, 8)\)

**Exercise #7:** Which of the following linear functions would *not* have an inverse that is also a function? Explain how you made your choice.

(1) \( y = x \)  
(2) \( 2y = x \)

(3) \( y = 2 \)  
(4) \( y = 5x - 1 \)
INVERSES OF LINEAR FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. The graph of a function and its inverse are always symmetric across which of the following lines?
   (1) \( y = 0 \)  
   (2) \( x = 0 \)  
   (3) \( y = x \)  
   (4) \( y = 1 \)

2. Which of the following represents the inverse of the linear function \( y = 3x - 24 \)?
   (1) \( y = \frac{1}{3}x + 8 \)  
   (2) \( y = \frac{1}{3}x - 8 \)  
   (3) \( y = -\frac{1}{3}x + 24 \)  
   (4) \( y = \frac{1}{3}x - \frac{1}{24} \)

3. If the \( y \)-intercept of a linear function is 8, then we know which of the following about its inverse?
   (1) Its \( y \)-intercept is \(-8\).  
   (2) Its \( x \)-intercept is 8.  
   (3) Its \( y \)-intercept is \( \frac{1}{8} \).  
   (4) Its \( x \)-intercept is \(-8\).

4. If both were plotted, which of the following linear functions would be parallel to its inverse? Explain your thinking.
   (1) \( y = 2x \)  
   (2) \( y = \frac{2}{3}x - 4 \)  
   (3) \( y = 5x - 1 \)  
   (4) \( y = x + 6 \)

5. Which of the following represents the equation of the inverse of \( y = \frac{4}{3}x + 24 \)?
   (1) \( y = -\frac{4}{3}x - 24 \)  
   (2) \( y = \frac{3}{4}x + 18 \)  
   (3) \( y = \frac{3}{4}x - 18 \)  
   (4) \( y = \frac{4}{3}x - 24 \)

6. Which of the following points lies on the inverse of \( y + 2 = 4(x - 1) \)?
   (1) \((2, -1)\)  
   (2) \((-1, 2)\)  
   (3) \(\left(\frac{1}{2}, 1\right)\)  
   (4) \((-2, 1)\)
7. A linear function is graphed below. Answer the following questions based on this graph.

(a) Write the equation of this linear function in \( y = mx + b \) form.

(b) Sketch a graph of the inverse of this function on the same grid.

(c) Write the equation of the inverse in \( y = mx + b \) form.

(d) What is the intersection point of this line with its inverse?

8. A car traveling at a constant speed of 58 miles per hour has a distance of \( y \)-miles from Poughkeepsie, NY, given by the equation \( y = 58x + 24 \), where \( x \) represents the time in hours that the car has been traveling.

(a) Find the equation of the inverse of this linear function in \( y = \frac{x-a}{b} \) form.

(b) Evaluate the function you found in part (a) for an input of \( x = 227 \).

(c) Give a physical interpretation of the answer you found in part (b). Consider what the input and output of the inverse represent in order to answer this question.

9. Given the general linear function \( y = mx + b \), find an equation for its inverse in terms of \( m \) and \( b \).
Functions expressed algebraically can sometimes be more complicated and involve different equations for different portions of their domains. These are known as piecewise functions (they come in pieces). If all of the pieces are linear, then they are known as piecewise linear functions.

**Exercise #1:** Consider the piecewise linear function given by the formula \( f(x) = \begin{cases} 
  x - 3 & \text{if } -3 \leq x < 0 \\
  \frac{1}{2}x + 4 & \text{if } 0 \leq x \leq 4 
\end{cases} \).

(a) Create a table of values below and graph the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
</tr>
</tbody>
</table>

(b) State the range of \( f \) using interval notation.

Not only should we be able to graph piecewise functions when we are given their equations, but we should also be able to translate the graphs of these functions into equations.

**Exercise #2:** The function \( f(x) \) is shown graphed below. Write a piecewise linear formula for the function. Be sure to specify both the formulas and the domain intervals over which they apply.
Piecewise equations can be challenging algebraically. Sometimes information that we find from them can be misleading or incorrect.

**Exercise #3:** Consider the piecewise linear function \( g(x) = \begin{cases} 5 - x & x < 2 \\ \frac{1}{2}x + 2 & x \geq 2 \end{cases} \).

(a) Determine the \( y \)-intercept of this function algebraically. Why can a function have only one \( y \)-intercept?

(b) Find the \( x \)-intercepts of each individual linear equation.

(c) Graph the piecewise linear function below.

(d) Why does your graph contradict the answers you found in part (b)?

(e) How can you resolve the fact that the algebra seems to contradict your graphical evidence of \( x \)-intercepts?

**Exercise #4:** For the piecewise linear function \( f(x) = \begin{cases} -2x + 10 & x \leq 0 \\ 5x - 1 & x > 0 \end{cases} \), find all solutions to the equation \( f(x) = 1 \) algebraically.
PIECEWISE LINEAR FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

For \( f(x) = \begin{cases} 
5x - 3 & x < -2 \\
x + 8 & -2 \leq x < 3 \\
x + 7 & x \geq 3 
\end{cases} \) answer the following questions.

(a) Evaluate each of the following by carefully applying the correct formula:

(i) \( f(2) \)  
(ii) \( f(-4) \)  
(iii) \( f(3) \)  
(iv) \( f(0) \)

(b) The three linear equations have \( y \)-intercepts of \(-3, 8 \) and \(7 \) respectively. Yet, a function can have only one \( y \)-intercept. Which of these is the \( y \)-intercept of this function? Explain how you made your choice.

(c) Calculate the average rate of change of \( f \) over the interval \(-3 \leq x \leq 9 \). Show the calculations that lead to your answer.

2. Determine the range of the function \( g(x) = \begin{cases} 
x + 4 & -2 \leq x \leq 2 \\
-\frac{3}{2}x + 9 & 2 < x \leq 6 
\end{cases} \) graphically.
3. Determine a piecewise linear equation for the function \( f(x) \) shown below. Be sure to specify not only the equations, but also the domain intervals over which they apply.

\[
f(x) = \begin{cases} 
-2 & 0 \leq x < 3 \\
3 & 3 \leq x < 5 \\
7 & 5 \leq x < 10 \\
5 & 10 \leq x \leq 12 
\end{cases}
\]

**REASONING**

4. Step functions are piecewise functions that are constants (horizontal lines) over each part of their domains. Graph the following step function.

5. Find all \( x \)-intercepts of the function \( g(x) = \begin{cases} 
2x + 8 & -5 \leq x < -1 \\
-\frac{1}{2}x - 4 & -1 \leq x < 1 \\
-4x + 10 & 1 \leq x \leq 4 
\end{cases} \) algebraically. Justify your work by showing your algebra. Be sure to check your answers versus the domain intervals to make sure each solution is valid.
Systems of equations, or more than one equation, arise frequently in mathematics. To **solve a system** means to find all sets of values that simultaneously make all equations true. Of special importance are systems of linear equations. You have solved them in your last two Common Core math courses, but we will add to their complexity in this lesson.

**Exercise #1:** Solve the following system of equations by: (a) substitution and (b) by elimination.

(a) \[3x + 2y = -9 \]
\[2x + y = -7 \]

(b) \[3x + 2y = -9 \]
\[2x + y = -7 \]

You should be very familiar with solving two-by-two systems of linear equations (two equations and two unknowns). In this lesson, we will extend the method of **elimination** to linear systems of three equations and three unknowns. These linear systems serve as the basis for a field of math known as **Linear Algebra**.

**Exercise #2:** Consider the three-by-three system of linear equations shown below. Each equation is numbered in this first exercise to help keep track of our manipulations.

(1) \[2x + y + z = 15 \]
(2) \[6x - 3y - z = 35 \]
(3) \[-4x + 4y - z = -14 \]

(a) The **addition property of equality** allows us to add two equations together to produce a third valid equation. Create a system by adding equations (1) and (2) and (1) and (3). Why is this an effective strategy in this case?

(b) Use this new two-by-two system to solve the three-by-three.
Just as with two by two systems, sometimes three-by-three systems need to be manipulated by the multiplication property of equality before we can eliminate any variables.

**Exercise #3:** Consider the system of equations shown below. Answer the following questions based on the system.

\[
\begin{align*}
4x + y - 3z &= -6 \\
-2x + 4y + 2z &= 38 \\
5x - y - 7z &= -19
\end{align*}
\]

(a) Which variable will be easiest to eliminate? Why? Use the multiplicative property of equality and elimination to reduce this system to a two-by-two system.

(b) Solve the two-by-two system from (a) and find the final solution to the three-by-three system.

**Exercise #4:** Solve the system of equations shown below. Show each step in your solution process.

\[
\begin{align*}
4x - 2y + 3z &= 23 \\
x + 5y - 3z &= -37 \\
-2x + y + 4z &= 27
\end{align*}
\]

These are challenging problems only because they are long. Be careful and you will be able to solve each one of these more complex systems.
Fluency

1. The sum of two numbers is 5 and the larger difference of the two numbers is 39. Find the two numbers by setting up a system of two equations with two unknowns and solving algebraically.

2. Algebraically, find the intersection points of the two lines whose equations are shown below.

   \[4x + 3y = -13\]
   \[y = 6x - 8\]

3. Show that \(x = 10, y = 4,\) and \(z = 7\) is a solution to the system below without solving the system formally.

   \[x + 2y + z = 25\]
   \[4x - y - 5z = 1\]
   \[-2x - y + 8z = 32\]

4. In the following system, the value of the constant \(c\) is unknown, but it is known that \(x = -8\) and \(y = 4\) are the \(x\) and \(y\) values that solve this system. Determine the value of \(c\). Show how you arrived at your answer.

   \[-5x + 2y + 3z = 81\]
   \[x - y + z = -1\]
   \[2x - y + cz = 35\]
5. Solve the following system of equations. Carefully show how you arrived at your answers.

\[ \begin{align*}
4x + 2y - z &= 21 \\
-x - 2y + 2z &= 13 \\
3x - 2y + 5z &= 70
\end{align*} \]

6. Algebraically solve the following system of equations. There are two variables that can be readily eliminated, but your answers will be the same no matter which you eliminate first.

\[ \begin{align*}
2x + 5y - z &= -35 \\
x - 3y + 4z &= 31 \\
-3x + 2y + 2z &= -23
\end{align*} \]

7. Algebraically solve the following system of equations. This system will take more manipulation because there are no variables with coefficients equal to 1.

\[ \begin{align*}
2x + 3y - 2z &= 33 \\
4x + 5y + 3z &= 54 \\
-6x - 2y - 8z &= -50
\end{align*} \]
UNIT #4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Lesson #1 – Integer Exponents
Lesson #2 – Rational Exponents
Lesson #3 – Exponential Function Basics
Lesson #4 – Finding Equations of Exponentials
Lesson #5 – The Method of Common Bases
Lesson #6 – Exponential Modeling with Percent Growth and Decay
Lesson #7 – Mindful Percent Manipulations
Lesson #8 – Introduction to Logarithms
Lesson #9 – Graphs of Logarithms
Lesson #10 – Logarithm Laws
Lesson #11 – Solving Exponential Equations Using Logarithms
Lesson #12 – The Number e and the Natural Logarithm
Lesson #13 – Compound Interest
Lesson #14 – Newton's Law of Cooling
We just finished our review of linear functions. Linear functions are those that grow by equal differences for equal intervals. In this unit we will concentrate on exponential functions which grow by equal factors for equal intervals. To understand exponential functions, we first need to understand exponents.

**Exercise #1:** The following sequence shows powers of 3 by repeatedly multiplying by 3. Fill in the missing blanks.

$$3^{-3} \quad 3^{-2} \quad 3^{-1} \quad 3^0 \quad 3^1 \quad 3^2 \quad 3^3$$

This pattern can be duplicated for any base raised to any integer exponent. Because of this we can now define positive, negative, and zero exponents in terms of multiplying the number 1 repeatedly or dividing the number 1 repeatedly.

<table>
<thead>
<tr>
<th>INTEGER EXPONENT DEFINITIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $n$ is any positive integer then:</td>
</tr>
<tr>
<td>1. $b^n = 1 \cdot b \cdot b \cdot \ldots \cdot b \cdot b$</td>
</tr>
<tr>
<td>n-times</td>
</tr>
<tr>
<td>2. $b^0 = 1$</td>
</tr>
<tr>
<td>3. $b^{-n} = \frac{1}{b \cdot b \cdot \ldots \cdot b \cdot b} = \frac{1}{b^n}$</td>
</tr>
<tr>
<td>n-times</td>
</tr>
</tbody>
</table>

**Exercise #2:** Given the exponential function $f(x) = 20(2)^x$ evaluate each of the following without using your calculator. Show the calculations that lead to your final answer.

(a) $f(2)$
(b) $f(0)$
(c) $f(-2)$

(d) When $x$ increases by 3, by what factor does $y$ increase? Explain your answer.
There are many basic **exponent properties or laws** that are critically important and that can be investigated using integer exponent examples. Two of the very important ones we will see next.

**Exercise #3:** For each of the following, write the product as a single exponential expression. Write (a) and (b) as extended products first (if necessary).

(a) \(3^2 \cdot 2^4\)  
(b) \(2^6 \cdot 2^2\)  
(c) \(2^n \cdot 2^n\)

It's clear why the exponent law that you generalized in part (c) works for positive integer exponents. But, does it also make sense within the context of our negative exponents?

**Exercise #4:** Consider now the product \(2^3 \cdot 2^{-1}\).

(a) Use the exponent law found in Exercise 3(c) to write this as a single exponential expression.  
(b) Evaluate \(2^3 \cdot 2^{-1}\) by first rewriting \(2^3\) and \(2^{-1}\) and then simplifying.

(c) Do your answers from (a) and (b) support the extension of the **Addition Property of Exponents** to negative powers as well? Explain.

Let's look at another important exponent property.

**Exercise #5:** For each of the following, write the exponential expression in the form \(3^x\). Write (a) and (b) as extended products first (if necessary).

(a) \((3^2)^3\)  
(b) \((3^4)^2\)  
(c) \((3^n)^n\)

Again, let's look at how the **Product Property of Exponents** still holds for negative exponents.

**Exercise #6:** Consider the expression \((3^{-2})^4\). Show this expression is equivalent to \(3^{-8}\) by first rewriting \(3^{-2}\) in fraction form.
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Write each of the following exponential expressions without the use of exponents such as we did in lesson Exercise #1.

(a) \(2^{-3}\) \(\times 2\)

(b) \(5^{-3}\) \(\times 5\)

(c) \(2^{-2}\)

(d) \(5^{-2}\)

(e) \(2^{-1}\)

(f) \(5^{-1}\)

(g) \(2^0\)

(h) \(5^0\)

(i) \(2^1\)

(j) \(5^1\)

(k) \(2^2\)

(l) \(5^2\)

(m) \(2^3\)

(n) \(5^3\)

2. Now let's go the other way around. For each of the following, determine the integer value of \(n\) that satisfies the equation. The first is done for you.

(a) \(2^n = \frac{1}{8}\)

\(2^n = \frac{1}{2^3}\)

\(n = -3\)

(b) \(4^n = 16\)

(c) \(3^n = \frac{1}{81}\)

(d) \(7^n = 1\)

\(2^n = \frac{1}{2^3}\)

\(n = -3\)

(e) \(5^n = \frac{1}{25}\)

(f) \(10^n = \frac{1}{10,000}\)

(g) \(13^n = 1\)

(h) \(2^n = \frac{1}{32}\)
3. Use the **Addition Property of Exponents** to simplify each expression. Then, find a final numerical answer *without* using your calculator.

   (a) \(2^{-5} \cdot 2^3 \cdot 2^4\)  

   (b) \(5^3 \cdot 5^7 \cdot 5^{-10}\)  

   (c) \(10^3 \cdot 10^{-7} \cdot 10^2\)

4. Use the **Product Property of Exponents** to simplify each exponential expression. You do not need to find a final numerical answer.

   (a) \((2^3)^4\)  

   (b) \((3^{-2})^2\)  

   (c) \(\left(5^2\right)^{-4} \cdot 2\)  

5. The exponential expression \(\left(\frac{1}{8}\right)^4\) is equivalent to which of the following? Explain your choice.

   (1) \(4^{-8}\)  

   (2) \(2^{-12}\)  

   (3) \(8^{-2}\)  

   (4) \(32^{-1}\)

**REASONING**

6. How can you use the fact that \(25^2 = 625\) to show that \(5^{-4} = \frac{1}{625}\)? Explain your process of thinking.

7. We've extended the two fundamental exponent properties to negative as well as positive integers. What would happen if we extended the **Product Exponent Property** to a fractional exponent like \(\frac{1}{2}\)? Let's play around with that idea.

   (a) Use the **Product Property of Exponents** to justify that \(\left(9^{\frac{1}{2}}\right)^2 = 9\).

   (b) What other number can you square that results in \(9\)? Hmm...
RATIONAL EXPONENTS
COMMON CORE ALGEBRA II

When you first learned about exponents, they were always positive integers, and just represented repeated multiplication. And then we had to go and introduce negative exponents, which really just represent repeated division. Today we will introduce rational (or fractional) exponents and extend your exponential knowledge that much further.

Exercise #1: Recall the Product Property of Exponents and use it to rewrite each of the following as a simplified exponential expression. There is no need to find a final numerical value.

(a) \( (2^1)^4 \)  
(b) \( (5^{-2})^5 \)  
(c) \( (3^7)^0 \)  
(d) \( (4^2)^{-2} \)

We will now use the Product Property to extend our understanding of exponents to include unit fraction exponents (those of the form \( \frac{1}{n} \) where \( n \) is a positive integer).

Exercise #2: Consider the expression \( 16^{\frac{1}{2}} \).

(a) Apply the Product Property to simplify \( \left(16^{\frac{1}{2}}\right)^2 \). What other number squared yields 16?

(b) You can now say that \( 16^{\frac{1}{2}} \) is equivalent to what more familiar quantity?

This is remarkable! An exponent of \( \frac{1}{2} \) is equivalent to a square root of a number!!!

Exercise #3: Test the equivalence of the \( \frac{1}{2} \) exponent to the square root by using your calculator to evaluate each of the following. Be careful in how you enter each expression.

(a) \( 25^{\frac{1}{2}} = \)  
(b) \( 81^{\frac{1}{2}} = \)  
(c) \( 100^{\frac{1}{2}} = \)

We can extend this now to all levels of roots, that is square roots, cubic roots, fourth roots, etcetera.

UNIT FRACTION EXPONENTS

For \( n \) given as a positive integer:  
\[ b^{\frac{1}{n}} = \sqrt[n]{b} \]
**Exercise #4:** Rewrite each of the following using roots instead of fractional exponents. Then, if necessary, evaluate using your calculator to guess and check to find the roots (don't use the generic root function). Check with your calculator.

(a) $125^{\frac{1}{3}}$ 
(b) $16^{\frac{1}{4}}$ 
(c) $9^{-\frac{1}{2}}$ 
(d) $32^{-\frac{1}{5}}$

We can now combine traditional integer powers with unit fractions in order interpret any exponent that is a *rational number*, i.e. the *ratio of two integers*. The next exercise will illustrate the thinking. Remember, we want our exponent properties to be consistent with the structure of the expression.

**Exercise #5:** Let's think about the expression $4^{\frac{1}{2}}$.

(a) Fill in the missing blank and then evaluate this expression:

$$4^{\frac{1}{2}} = (\phantom{0})^{\frac{1}{2}}$$

(b) Fill in the missing blank and then evaluate this expression:

$$4^{\frac{1}{2}} = (\phantom{0})^{3}$$

(c) Verify both (a) and (b) using your calculator.

(d) Evaluate $27^{\frac{1}{3}}$ without your calculator. Show your thinking. Verify with your calculator.

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**RATIONAL EXPONENT CONNECTION TO ROOTS**

For the rational number $\frac{m}{n}$ we define $b^{\frac{m}{n}}$ to be: $\sqrt[n]{b^m}$ or $(\sqrt[n]{b})^m$.

**Exercise #6:** Evaluate each of the following exponential expressions involving rational exponents without the use of your calculator. Show your work. Then, check your final answers with the calculator.

(a) $16^{\frac{1}{4}}$ 
(b) $25^{\frac{3}{2}}$ 
(c) $8^{\frac{2}{3}}$
RATIONAL EXPONENTS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Rewrite the following as equivalent roots and then evaluate as many as possible without your calculator.
   (a) \(36^{\frac{1}{2}}\)  
   (b) \(27^{\frac{1}{3}}\) 
   (c) \(32^{\frac{1}{5}}\)  
   (d) \(100^{-\frac{1}{2}}\) 
   (e) \(625^{\frac{1}{4}}\)  
   (f) \(49^{\frac{1}{2}}\) 
   (g) \(81^{-\frac{1}{4}}\)  
   (h) \(343^{\frac{1}{3}}\)

2. Evaluate each of the following by considering the root and power indicated by the exponent. Do as many as possible without your calculator.
   (a) \(8^{\frac{2}{3}}\)  
   (b) \(4^{\frac{3}{2}}\) 
   (c) \(16^{\frac{3}{4}}\)  
   (d) \(81^{\frac{5}{4}}\) 
   (e) \(4^{-\frac{5}{2}}\)  
   (f) \(128^{\frac{3}{7}}\) 
   (g) \(625^{\frac{3}{4}}\)  
   (h) \(243^{\frac{3}{5}}\)

3. Given the function \(f(x) = 5(x + 4)^{\frac{3}{2}}\), which of the following represents its \(y\)-intercept?
   (1) 40  
   (2) 20  
   (3) 4  
   (4) 30
4. Which of the following is equivalent to \( x^{-\frac{1}{2}} \)?

(1) \(-\frac{1}{2}x\)  
(3) \(\frac{1}{\sqrt{x}}\)  
(2) \(-\sqrt{x}\)  
(4) \(-\frac{1}{2x}\)

5. Written without fractional or negative exponents, \( x^{-\frac{1}{2}} \) is equal to

(1) \(-\frac{3x}{2}\)  
(3) \(\frac{1}{\sqrt{x^3}}\)  
(2) \(\frac{1}{\sqrt{x^2}}\)  
(4) \(-\frac{1}{\sqrt{x}}\)

6. Which of the following is not equivalent to \( 16^{\frac{1}{2}} \)?

(1) \(\sqrt{4096}\)  
(3) 64  
(2) \(8^3\)  
(4) \(\sqrt[3]{16^3}\)

**REASONING**

7. Marlene claims that the square root of a cube root is a sixth root? Is she correct? To start, try rewriting the expression below in terms of fractional exponents. Then apply the **Product Property of Exponents**.

\[ \sqrt[3]{\sqrt{a}} \]

8. We should know that \( \sqrt[3]{8} = 2 \). To see how this is equivalent to \( 8^{\frac{1}{3}} = 2 \) we can solve the equation \( 8^n = 2 \). To do this, we can rewrite the equation as:

\[ (2^3)^n = 2^1 \]

How can we now use this equation to see that \( 8^{\frac{1}{3}} = 2 \)?
You studied exponential functions extensively in Common Core Algebra I. Today's lesson will review many of the basic components of their graphs and behavior. Exponential functions, those whose exponents are variable, are extremely important in mathematics, science, and engineering.

### Basic Exponential Functions

\[ y = b^x \quad \text{where } b > 0 \text{ and } b \neq 1 \]

#### Exercise #1:
Consider the function \( y = 2^x \). Fill in the table below without using your calculator and then sketch the graph on the grid provided.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 2^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
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<td>-2</td>
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<td>-1</td>
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<td>2</td>
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<tr>
<td>3</td>
<td></td>
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</tbody>
</table>

#### Exercise #2:
Now consider the function \( y = \left(\frac{1}{2}\right)^x \). Using your calculator to help you, fill out the table below and sketch the graph on the axes provided.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \left(\frac{1}{2}\right)^x )</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>
**Exercise #3:** Based on the graphs and behavior you saw in Exercises #1 and #2, state the domain and range for an exponential function of the form $y = b^x$.

Domain (input set):  
Range (output set):

**Exercise #4:** Are exponential functions one-to-one? How can you tell? What does this tell you about their inverses?

**Exercise #5:** Now consider the function $y = 7(3)^x$.

(a) Determine the $y$-intercept of this function algebraically. Justify your answer.

(b) Does the exponential function increase or decrease? Explain your choice.

(c) Create a rough sketch of this function, labeling its $y$-intercept.

**Exercise #6:** Consider the function $y = \left(\frac{1}{3}\right)^x + 4$.

(a) How does this function’s graph compare to that of $y = \left(\frac{1}{3}\right)^x$? What does adding 4 do to a function's graph?

(b) Determine this graph’s $y$-intercept algebraically. Justify your answer.

(c) Create a rough sketch of this function, labeling its $y$-intercept.
EXPONENTIAL FUNCTION BASICS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following represents an exponential function?
   (1) \( y = 3x - 7 \)  (3) \( y = 3(7)^x \)
   (2) \( y = 7x^3 \)  (4) \( y = 3x^2 + 7 \)

2. If \( f(x) = 6(9)^x \) then \( f\left(\frac{1}{2}\right) = ? \) (Remember what we just learned about fractional exponents and do without a calculator.)
   (1) \( \frac{7}{2} \)  (3) 27
   (2) 18  (4) \( \frac{15}{2} \)

3. If \( h(x) = 3^x \) and \( g(x) = 5x - 7 \) then \( h(g(2)) = \)
   (1) 18  (3) 38
   (2) 12  (4) 27

4. Which of the following equations could describe the graph shown below?
   (1) \( y = x^2 + 1 \)  (3) \( y = -2x + 1 \)
   (2) \( y = \left(\frac{2}{3}\right)^x \)  (4) \( y = 4^x \)

5. Which of the following equations represents the graph shown?
   (1) \( y = 5^x \)  (3) \( y = \left(\frac{1}{2}\right)^x + 2 \)
   (2) \( y = 4^x + 1 \)  (4) \( y = 3^x + 2 \)
6. Sketch graphs of the equations shown below on the axes given. Label the $y$-intercepts of each graph.

(a) $y = 18 \left( \frac{1}{3} \right)^x$

(b) $y = 25 \left( 4 \right)^x$

![Graphs of exponential functions]

**APPLICATION**

7. The Fahrenheit temperature of a cup of coffee, $F$, starts at a temperature of $185^\circ F$. It cools down according to the exponential function $F(m) = 113 \left( \frac{1}{2} \right)^{m/20} + 72$, where $m$ is the number minutes it has been cooling.

(a) How do you interpret the statement that $F(60) = 86$?

(b) Determine the temperature of the coffee after one day using your calculator. What do you think this temperature represents about the physical situation?

**REASONING**

8. The graph below shows two exponential functions, with real number constants $a$, $b$, $c$, and $d$. Given the graphs, only one pair of the constants shown below could be equal in value. Determine which pair could be equal and explain your reasoning.

- $b$ and $d$
- $a$ and $b$
- $a$ and $c$

![Graphs of exponential functions]

9. Explain why the equation below can have no real solutions. If you need to, graph both sides of the equation using your calculator to visualize the reason.

$3^x + 5 = 2$
Finding Equations of Exponential Functions

COMMON CORE ALGEBRA II

One of the skills that you acquired in Common Core Algebra I was the ability to write equations of exponential functions if you had information about the starting value and base (multiplier or growth constant). Let's review a very basic problem.

Exercise #1: An exponential function of the form \( f(x) = a(b)^x \) is presented in the table below. Determine the values of \( a \) and \( b \) and explain your reasoning.

\[
\begin{array}{c|cccc}
 x & 0 & 1 & 2 & 3 \\
 f(x) & 5 & 15 & 45 & 135 \\
\end{array}
\]

\[ a = \boxed{\rule{50pt}{0pt}} \]
\[ b = \boxed{\rule{50pt}{0pt}} \]

Final Equation: \[ \boxed{\rule{50pt}{0pt}} \] Description:

Finding an exponential equation becomes much more challenging if we do not have output values for inputs that are increasing by unit values (increasing by 1 unit at a time). Let's start with a basic problem.

Exercise #2: For an exponential function of the form \( f(x) = a(b)^x \), it is known that \( f(0) = 8 \) and \( f(3) = 1000 \).

(a) Use the fact that \( f(0) = 8 \) to determine the value of \( a \). Show your thinking.

(b) Use your answer from part (a) and the fact that \( f(3) = 1000 \) to set up an equation to solve for \( b \). You will solve for \( b \) in part (c).

(c) Solve for the value of \( b \) using properties of exponents.

(d) Determine the value of \( f(2) \)

Exercise #3: An exponential function exists such that \( f(4) = 3 \) and \( f(6) = 48 \), which of the following must be the value of its base? Explain or illustrate your thinking.

(1) \( b = 16 \)  
(2) \( b = 2 \)  
(3) \( b = 6 \)  
(4) \( b = 4 \)
Now, let's work with the most generic type of problem. Just like with lines, any two (non-vertically aligned) points will uniquely determine the equation of an exponential function.

**Exercise #4:** An exponential function of the form \( y = a(b)^x \) passes through the points \((2, 36)\) and \((5, 121.5)\).

(a) By substituting these two points into the general form of the exponential, create a **system of equations** in the constants \(a\) and \(b\).

(b) Divide these two equations to eliminate the constant \(a\). Recall that when dividing to like bases, you subtract their exponents.

(c) Solve the resulting equation from (b) for the base, \(b\).

(d) Use your value from (c) to determine the value of \(a\). State the final equation.

Let's now get some practice on this with a decreasing exponential function.

**Exercise #5:** Find the equation of the exponential function shown graphed below. Be careful in terms of your exponent manipulation. State your final answer in the form \(y = a(b)^x\).

![Graph of an exponential function](image)

**Exercise #6:** A bacterial colony is growing at an exponential rate. It is known that after 4 hours, its population is at 98 bacteria and after 9 hours it is 189 bacteria. Determine an equation in \(y = a(b)^x\) form that models the population, \(y\), as a function of the number of hours, \(x\). At what percent rate is the population growing per hour?
**FINDING EQUATIONS OF EXPONENTIAL FUNCTIONS**  
**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

1. For each of the following coordinate pairs, find the equation of the exponential function, in the form \( y = a(b)^x \) that passes through the pair. Show the work that you use to arrive at your answer.

   (a) \((0,10) \) and \((3, 80)\)

   (b) \((0,180) \) and \((2, 80)\)

2. For each of the following coordinate pairs, find the equation of the exponential function, in the form \( y = a(b)^x \) that passes through the pair. Show the work that you use to arrive at your answer.

   (a) \((2,192) \) and \((5,12288)\)

   (b) \((1,192) \) and \((5, 60.75)\)

3. Each of the previous problems had values of \(a\) and \(b\) that were rational numbers. They do not need not be. Find the equation for an exponential function that passes through the points \((2,14)\) and \((7,205)\) in \( y = a(b)^x \) form. When you find the value of \(b\) do not round your answer before you find \(a\). Then, find both to the nearest *hundredth* and give the final equation. Check to see if the points fall on the curve.
APPLICATIONS

4. A population of koi goldfish in a pond was measured over time. In the year 2002, the population was recorded as 380 and in 2006 it was 517. Given that \( y \) is the population of fish and \( x \) is the number of years since 2000, do the following:

(a) Represent the information in this problem as two coordinate points.

(b) Determine a linear function in the form \( y = mx + b \) that passes through these two points. Don’t round the linear parameters \( m \) and \( b \).

(c) Determine an exponential function of the form \( y = a(b)^x \) that passes through these two points. Round \( b \) to the nearest hundredth and \( a \) to the nearest tenth.

(d) Which model predicts a larger population of fish in the year 2000? Justify your work.

5. Engineers are draining a water reservoir until its depth is only 10 feet. The depth decreases exponentially as shown in the graph below. The engineers measure the depth after 1 hour to be 64 feet and after 4 hours to be 28 feet. Develop an exponential equation in \( y = a(b)^x \) to predict the depth as a function of hours draining. Round \( a \) to the nearest integer and \( b \) to the nearest hundredth. Then, graph the horizontal line \( y = 10 \) and find its intersection to determine the time, to the nearest tenth of an hour, when the reservoir will reach a depth of 10 feet.
The Method of Common Bases

There are very few algebraic techniques that do not involve technology to solve equations that contain exponential expressions. In this lesson we will look at one of the few, known as The Method of Common Bases.

Exercise #1: Solve each of the following simple exponential equations by writing each side of the equation using a common base.

(a) \(2^x = 16\)  
(b) \(3^x = 27\)  
(c) \(5^x = \frac{1}{25}\)  
(d) \(16^x = 4\)

In each of these cases, even the last, more challenging one, we could manipulate the right-hand side of the equation so that it shared a common base with the left-hand side of the equation. We can exploit this fact by manipulating both sides so that they have a common base. First, though, we need to review an exponent law.

Exercise #2: Simplify each of the following exponential expressions.

(a) \((2^3)^x\)  
(b) \((3^2)^{4x}\)  
(c) \((5^{-1})^{3x-7}\)  
(d) \((4^{-3})^{1-x^2}\)

Exercise #3: Solve each of the following equations by finding a common base for each side.

(a) \(8^x = 32\)  
(b) \(9^{2x+1} = 27\)  
(c) \(125^x = \left(\frac{1}{25}\right)^{4-x}\)

Exercise #4: Which of the following represents the solution set to the equation \(2^{x^2-3} = 64\)?

1. \(\{\pm 3\}\)
2. \(\{0, 3\}\)
3. \(\{\pm \sqrt{11}\}\)
4. \(\{\pm \sqrt{35}\}\)
This technique can be used in any situation where all bases involved can be written with a common base. In a practical sense, this is rather rare. Yet, these types of algebraic manipulations help us see the structure in exponential expressions. Try to tackle the next, more challenging, problem.

**Exercise #5:** Two exponential curves, \( y = 4^{\frac{x^5}{2}} \) and \( y = \left(\frac{1}{2}\right)^{2x+1} \) are shown below. They intersect at point A. A rectangle has one vertex at the origin and the other at A as shown. We want to find its area.

(a) Fundamentally, what do we need to know about a rectangle to find its area?

(b) How would knowing the coordinates of point A help us find the area?

(c) Find the area of the rectangle algebraically using the Method of Common Bases. Show your work carefully.

**Exercise #6:** At what \( x \) coordinate will the graph of \( y = 25^{x-a} \) intersect the graph of \( y = \left(\frac{1}{125}\right)^{3x+1} \)? Show the work that leads to your choice.

(1) \( x = \frac{5a-1}{3} \)  
(3) \( x = \frac{-2a+1}{5} \)

(2) \( x = \frac{2a-3}{11} \)  
(4) \( x = \frac{5a+3}{2} \)
THE METHOD OF COMMON BASES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following exponential equations using the Method of Common Bases. Each equation will result in a linear equation with one solution. Check your answers.

(a) \( 3^{2x-5} = 9 \)  

(b) \( 2^{3x+7} = 16 \)  

(c) \( 5^{4x-5} = \frac{1}{125} \)

(d) \( 8^x = 4^{2x+1} \)

(e) \( 216^{x-2} = \left(\frac{1}{1296}\right)^{3x-2} \)

(f) \( \left(\frac{1}{25}\right)^{x+15} = 3125^{\frac{3}{x-1}} \)

2. *Algebraically* determine the intersection point of the two exponential functions shown below. Recall that most systems of equations are solved by substitution.

\[ y = 8^{x-1} \quad \text{and} \quad y = 4^{2x-3} \]

3. *Algebraically* determine the zeroes of the exponential function \( f(x) = 2^{2x-9} - 32 \). Recall that the reason it is known as a zero is because the output is zero.
APPLICATIONS

4. One hundred must be raised to what power in order to be equal to a million cubed? Solve this problem using the Method of Common Bases. Show the algebra you do to find your solution.

5. The exponential function \( y = \left( \frac{1}{25} \right)^{\frac{x-2}{5}} - 10 \) is shown graphed along with the horizontal line \( y = 115 \). Their intersection point is \((a, 115)\). Use the Method of Common Bases to find the value of \( a \). Show your work.

REASONING

6. The Method of Common Bases works because exponential functions are one-to-one, i.e. if the outputs are the same, then the inputs must also be the same. This is what allows us to say that if \( 2^x = 2^3 \), then \( x \) must be equal to 3. But it doesn't always work out so easily.

If \( x^2 = 5^2 \), can we say that \( x \) must be 5? Could it be anything else? Why does this not work out as easily as the exponential case?
Exponential Modeling with Percent Growth and Decay

Common Core Algebra II

Exponential functions are very important in modeling a variety of real world phenomena because certain things either increase or decrease by fixed percentages over given units of time. You looked at this in Common Core Algebra I and in this lesson we will review much of what you saw.

Exercise #1: Suppose that you deposit money into a savings account that receives 5% interest per year on the amount of money that is in the account for that year. Assume that you deposit $400 into the account initially.

(a) How much will the savings account increase by over the course of the year?
(b) How much money is in the account at the end of the year?

(c) By what single number could you have multiplied the $400 by in order to calculate your answer in part (b)?
(d) Using your answer from part (c), determine the amount of money in the account after 2 and 10 years. Round all answers to the nearest cent when needed.

(e) Give an equation for the amount in the savings account $S(t)$ as a function of the number of years since the $400 was invested.
(f) Using a table on your calculator determine, to the nearest year, how long it will take for the initial investment of $400 to double. Provide evidence to support your answer.

The thinking process from Exercise #1 can be generalized to any situation where a quantity is increased by a fixed percentage over a fixed interval of time. This pattern is summarized below:

**Increasing Exponential Models**

If quantity $Q$ is known to increase by a fixed percentage $p$, in decimal form, then $Q$ can be modeled by

$$Q(t) = Q_0 (1 + p)^t$$

where $Q_0$ represents the amount of $Q$ present at $t = 0$ and $t$ represents time.

Exercise #2: Which of the following gives the savings $S$ in an account if $250 was invested at an interest rate of 3% per year?

(1) $S = 250(4)^t$
(2) $S = 250(1.03)^t$
(3) $S = (1.03)^t + 250$
(4) $S = 250(1.3)^t$
Decreasing exponentials are developed in the same way, but have the percent subtracted, rather than added, to the base of 100%. Just remember, you are ultimately multiplying by the percent of the original that you will have after the time period elapses.

**Exercise #3:** State the multiplier (base) you would need to multiply by in order to decrease a quantity by the given percent listed.

(a) 10%  (b) 2%  (c) 25%  (d) 0.5%

---

**DECREASING EXPONENTIAL MODELS**

If quantity $Q$ is known to decrease by a fixed percentage $p$, in decimal form, then $Q$ can be modeled by

$$Q(t) = Q_0(1 - p)^t$$

where $Q_0$ represents the amount of $Q$ present at $t = 0$ and $t$ represents time.

---

**Exercise #4:** If the population of a town is decreasing by 4% per year and started with 12,500 residents, which of the following is its projected population in 10 years? Show the exponential model you use to solve this problem.

(1) 9,230  (3) 18,503  (2) 76  (4) 8,310

---

**Exercise #5:** The stock price of WindpowerInc is increasing at a rate of 4% per week. Its initial value was $20 per share. On the other hand, the stock price in GerbilEnergy is crashing (losing value) at a rate of 11% per week. If its price was $120 per share when Windpower was at $20, after how many weeks will the stock prices be the same? Model both stock prices using exponential functions. Then, find when the stock prices will be equal graphically. Draw a well labeled graph to justify your solution.
EXPONENTIAL MODELING WITH PERCENT GROWTH AND DECAY

COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. If $130 is invested in a savings account that earns 4% interest per year, which of the following is closest to the amount in the account at the end of 10 years?

   - (1) $218
   - (2) $192
   - (3) $168
   - (4) $324

2. A population of 50 fruit flies is increasing at a rate of 6% per day. Which of the following is closest to the number of days it will take for the fruit fly population to double?

   - (1) 18
   - (2) 6
   - (3) 12
   - (4) 28

3. If a radioactive substance is quickly decaying at a rate of 13% per hour approximately how much of a 200 pound sample remains after one day?

   - (1) 7.1 pounds
   - (2) 2.3 pounds
   - (3) 25.6 pounds
   - (4) 15.6 pounds

4. A population of llamas stranded on a dessert island is decreasing due to a food shortage by 6% per year. If the population of llamas started out at 350, how many are left on the island 10 years later?

   - (1) 257
   - (2) 58
   - (3) 102
   - (4) 189

5. Which of the following equations would model a population with an initial size of 625 that is growing at an annual rate of 8.5%?

   - (1) \( P = 625(8.5)^t \)
   - (2) \( P = 625(1.085)^t \)
   - (3) \( P = 1.085^t + 625 \)
   - (4) \( P = 8.5t^2 + 625 \)

6. The acceleration of an object falling through the air will decrease at a rate of 15% per second due to air resistance. If the initial acceleration due to gravity is 9.8 meters per second per second, which of the following equations best models the acceleration \( t \) seconds after the object begins falling?

   - (1) \( a = 15 - 9.8t^2 \)
   - (2) \( a = \frac{9.8}{15t} \)
   - (3) \( a = 9.8(1.15)^t \)
   - (4) \( a = 9.8(0.85)^t \)
7. Red Hook has a population of 6,200 people and is growing at a rate of 8% per year. Rhinebeck has a population of 8,750 and is growing at a rate of 6% per year. In how many years, to the nearest year, will Red Hook have a greater population than Rhinebeck? Show the equation or inequality you are solving and solve it graphically.

8. A warm glass of water, initially at 120 degrees Fahrenheit, is placed in a refrigerator at 34 degrees Fahrenheit and its temperature is seen to decrease according to the exponential function

\[ T(h) = 86(0.83)^h + 34 \]

(a) Verify that the temperature starts at 120 degrees Fahrenheit by evaluating \( T(0) \).

(b) Using your calculator, sketch a graph of \( T \) below for all values of \( h \) on the interval \( 0 \leq h \leq 24 \). Be sure to label your \( y \)-axis and \( y \)-intercept.

(c) After how many hours will the temperature be at 50 degrees Fahrenheit? State your answer to the nearest hundredth of an hour. Illustrate your answer on the graph you drew in (b).

**REASONING**

9. Percents combine in strange ways that don't seem to make sense at first. It would seem that if a population grows by 5% per year for 10 years, then it should grow in total by 50% over a decade. But this isn't true. Start with a population of 100. If it grows at 5% per year for 10 years, what is its population after 10 years? What percent growth does this represent?
Percents and phenomena that grow at a constant percent rate can be challenging, to say the least. This is due to the fact that, unlike linear phenomena, the growth rate indicates a constant multiplier effect instead of a constant additive effect (linear). Because constant percent growth is so common in everyday life (not to mention in science, business, and other fields), it's good to be able to mindfully manipulate percents.

**Exercise #1:** A population of wombats is growing at a constant percent rate. If the population on January 1st is 1027 and a year later is 1079, what is its yearly percent growth rate to the nearest tenth of a percent?

**Exercise #2:** Now let's try to determine what the percent growth in wombat population will be over a decade of time. We will assume that the rounded percent increase found in Exercise #1 continues for the next decade.

(a) After 10 years, what will we have multiplied the original population by, rounded to the nearest hundredth? Show the calculation.

(b) Using your answer from (a), what is the decade percent growth rate?

**Exercise #3:** Let's stick with our wombats from Exercise #1. Assuming their growth rate is constant over time, what is their monthly growth rate to the nearest tenth of a percent? Assume a constant sized month.

**Exercise #4:** If a population was growing at a constant rate of 22% every 5 years, then what is its percent growth rate over at 2 year time span? Round to the nearest tenth of a percent.

(a) First, give an expression that will calculate the single year (or yearly) percent growth rate based on the fact that the population grew 22% in 5 years.

(b) Now use this expression to calculate the percent growth over 2 years.
**Exercise #5:** World oil reserves (the amount of oil unused in the ground) are depleting at a constant 2% per year. We would like to determine what the percent decline will be over the next 20 years based on this 2% yearly decline.

(a) Write and evaluate an expression for what we would multiply the initial amount of oil by after 20 years.

(b) Use your answer to (a) to determine the percent decline after 20 years. Be careful! Round to the nearest percent.

**Exercise #6:** A radioactive substance’s half-life is the amount of time needed for half (or 50%) of the substance to decay. Let’s say we have a radioactive substance with a half-life of 20 years.

(a) What percent of the substance would be radioactive after 40 years?

(b) What percent of the substance would be radioactive after only 10 years? Round to the nearest tenth of a percent.

(c) What percent of the substance would be radioactive after only 5 years? Round to the nearest tenth of a percent.
MINDFUL MANIPULATION OF PERCENTS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. A quantity is growing at a constant 3% yearly rate. Which of the following would be its percent growth after 15 years?

   (1) 45%  
   (2) 52%  
   (3) 56%  
   (4) 63%

2. If a credit card company charges 13.5% yearly interest, which of the following calculations would be used in the process of calculating the monthly interest rate?

   (1) \( \frac{0.135}{12} \)  
   (2) \( \frac{1.135}{12} \)  
   (3) \( (1.135)^{12} \)  
   (4) \( (1.135)^{\frac{1}{12}} \)

3. The county debt is growing at an annual rate of 3.5%. What percent rate is it growing at per 2 years? Per 5 years? Per decade? Show the calculations that lead to each answer. Round each to the nearest tenth of a percent.

4. A population of llamas is growing at a constant yearly rate of 6%. At what rate is the llama population growing per month? Please assume all months are equally sized and that there are 12 of these per year. Round to the nearest tenth of a percent.
5. Shana is trying to increase the number of calories she burns by 5% per day. By what percent is she trying to increase per week? Round to the nearest tenth of a percent.

6. If a bank account doubles in size every 5 years, then by what percent does it grow after only 3 years? Round to the nearest tenth of a percent. Hint: First write an expression that would calculate its growth rate after a single year.

7. An object’s speed decreases by 5% for each minute that it is slowing down. Which of the following is closest to the percent that its speed will decrease over half-an hour?

   (1) 21%  
   (2) 79%  
   (3) 48%  
   (4) 150%

8. Over the last 10 years, the price of corn has decreased by 25% per bushel.

   (a) Assuming a steady percent decrease, by what percent does it decrease each year? Round to the nearest tenth of a percent.

   (b) Assuming this percent continues, by what percent will the price of corn decrease by after 50 years? Show the calculation that leads to your answer. Round to the nearest percent.
INTRODUCTION TO LOGARITHMS
COMMON CORE ALGEBRA II

Exponential functions are of such importance to mathematics that their inverses, functions that “reverse” their action, are important themselves. These functions, known as logarithms, will be introduced in this lesson.

**Exercise #1:** The function \( f(x) = 2^x \) is shown graphed on the axes below along with its table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2^x )</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>(1)</td>
<td>(2)</td>
<td>(4)</td>
<td>(8)</td>
</tr>
</tbody>
</table>

(a) Is this function one-to-one? Explain your answer.

(b) Based on your answer from part (a), what must be true about the inverse of this function?

(c) Create a table of values below for the inverse of \( f(x) = 2^x \) and plot this graph on the axes given.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f^{-1}(x) )</th>
</tr>
</thead>
</table>

(d) What would be the first step to find an equation for this inverse algebraically? Write this step down and then stop.

Defining Logarithmic Functions – The function \( y = \log_b x \) is the name we give the inverse of \( y = b^x \). For example, \( y = \log_2 x \) is the inverse of \( y = 2^x \). Based on Exercise #1(d), we can write an **equivalent exponential equation** for each logarithm as follows:

\[ y = \log_b x \text{ is the same as } b^y = x \]

Based on this, we see that a logarithm gives as its output (y-value) the exponent we must raise \( b \) to in order to produce its input (x-value).
**Exercise #2:** Evaluate the following logarithms. If needed, write an equivalent exponential equation. Do as many as possible without the use of your calculator.

(a) \( \log_2 8 \)  
(b) \( \log_4 16 \)  
(c) \( \log_5 625 \)  
(d) \( \log_{10} 100,000 \)

(e) \( \log_6 \left( \frac{1}{36} \right) \)  
(f) \( \log_2 \left( \frac{1}{16} \right) \)  
(g) \( \log_5 \sqrt{5} \)  
(h) \( \log_3 \sqrt[3]{9} \)

It is critically important to understand that logarithms **give exponents as their outputs**. We will be working for multiple lessons on logarithms and a basic understanding of their inputs and outputs is critical.

**Exercise #3:** If the function \( y = \log_2 (x + 8) + 9 \) was graphed in the coordinate plane, which of the following would represent its \( y \)-intercept?

1. 12  
2. 13  
3. 8  
4. 9

**Exercise #4:** Between which two consecutive integers must \( \log_3 40 \) lie?

1. 1 and 2  
2. 2 and 3  
3. 3 and 4  
4. 4 and 5

**Calculator Use and Logarithms** – Most calculators only have two logarithms that they can evaluate directly. One of them, \( \log_{10} x \), is so common that it is actually called the **common log** and typically is written without the base 10.

\[
\log x = \log_{10} x \quad (\text{The Common Log})
\]

**Exercise #5:** Evaluate each of the following using your calculator.

(a) \( \log 100 \)  
(b) \( \log \left( \frac{1}{1000} \right) \)  
(c) \( \log \sqrt{10} \)
INTRODUCTION TO LOGARITHMS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is equivalent to \( y = \log_b x \)?
   
   \[
   \begin{align*}
   (1) \quad y &= x^7 \\
   (2) \quad x &= y^7 \\
   (3) \quad x &= 7^y \\
   (4) \quad y &= x^{1/7}
   \end{align*}
   \]

2. If the graph of \( y = 6^x \) is reflected across the line \( y = x \) then the resulting curve has an equation of
   
   \[
   \begin{align*}
   (1) \quad y &= -6^x \\
   (2) \quad y &= \log_6 x \\
   (3) \quad x &= \log_6 y \\
   (4) \quad x &= y^6
   \end{align*}
   \]

3. The value of \( \log_5 167 \) is closest to which of the following? Hint – guess and check the answers.
   
   \[
   \begin{align*}
   (1) \quad 2.67 \\
   (2) \quad 1.98 \\
   (3) \quad 4.58 \\
   (4) \quad 3.18
   \end{align*}
   \]

4. Which of the following represents the \( y \)-intercept of the function \( y = \log(x + 1000) - 8 \)?
   
   \[
   \begin{align*}
   (1) \quad -8 \\
   (2) \quad -5 \\
   (3) \quad 3 \\
   (4) \quad 5
   \end{align*}
   \]

5. Determine the value for each of the following logarithms. (Easy)
   
   \[
   \begin{align*}
   (a) \quad \log_2 32 \\
   (b) \quad \log_7 49 \\
   (c) \quad \log_3 6561 \\
   (d) \quad \log_4 1024
   \end{align*}
   \]

6. Determine the value for each of the following logarithms. (Medium)
   
   \[
   \begin{align*}
   (a) \quad \log_2 \left(\frac{1}{64}\right) \\
   (b) \quad \log_3 (1) \\
   (c) \quad \log_5 \left(\frac{1}{25}\right) \\
   (d) \quad \log_9 \left(\frac{1}{343}\right)
   \end{align*}
   \]
7. Determine the value for each of the following logarithms. Each of these will have non-integer, fractional answers. (Difficult)

(a) $\log_4 2$  
(b) $\log_4 8$  
(c) $\log_5 \sqrt[5]{5}$  
(d) $\log_2 \sqrt[4]{4}$

8. Between what two consecutive integers must the value of $\log_4 7342$ lie? Justify your answer.

9. Between what two consecutive integers must the value of $\log_5 \left(\frac{1}{500}\right)$ lie? Justify your answer.

APPLICATIONS

10. In chemistry, the pH of a solution is defined by the equation $\text{pH} = -\log(H)$ where $H$ represents the concentration of hydrogen ions in the solution. Any solution with a pH less than 7 is considered acidic and any solution with a pH greater than 7 is considered basic. Fill in the table below. Round your pH’s to the nearest tenth of a unit.

<table>
<thead>
<tr>
<th>Substance</th>
<th>Concentration of Hydrogen</th>
<th>pH</th>
<th>Basic or Acidic?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milk</td>
<td>$1.6 \times 10^{-7}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coffee</td>
<td>$1.3 \times 10^{-5}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bleach</td>
<td>$2.5 \times 10^{-13}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lemmon Juice</td>
<td>$7.9 \times 10^{-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rain</td>
<td>$1.6 \times 10^{-6}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

REASONING

11. Can the value of $\log_2 (-4)$ be found? What about the value of $\log_2 0$? Why or why not? What does this tell you about the domain of $\log_b x$?
The vast majority of logarithms that are used in the real world have bases greater than one; the pH scale that we
saw on the last homework assignment is a good example. In this lesson we will further explore graphs of these
logarithms, including their construction, transformations, and domains and ranges.

**Exercise #1:** Consider the logarithmic function \( y = \log_3 x \) and its inverse \( y = 3^x \).

(a) Construct a table of values for \( y = 3^x \) and then use this to construct a table of values for the function \( y = \log_3 x \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -2 & -1 & 0 & 1 & 2 \\
\hline
y = 3^x & & & & & \\
\hline
x & & & & & \\
y = \log_3 x & & & & & \\
\hline
\end{array}\
\]

(b) Graph \( y = 3^x \) and \( y = \log_3 x \) on the grid given. Label with equations.

(c) State the natural domain and range of \( y = 3^x \) and \( y = \log_3 x \).

\[
\begin{align*}
y = 3^x & \quad \text{Domain: } (0, \infty) \\
y = \log_3 x & \quad \text{Domain: } (0, \infty) \\
& \quad \text{Range: } (-\infty, \infty) \\
& \quad \text{Range: } (-\infty, \infty)
\end{align*}\
\]

**Exercise #2:** Using your calculator, sketch the graph of \( y = \log_{10} x \) on the axes below. Label the x-intercept.
State the domain and range of \( y = \log_{10} x \).

\[
\begin{align*}
\text{Domain: } & \\
\text{Range: } & \\
\end{align*}\
\]
**Exercise #3:** Which of the following equations describes the graph shown below? Show or explain how you made your choice.

1. \( y = \log_3(x+2) - 1 \)
2. \( y = \log_2(x-3) + 1 \)
3. \( y = \log_2(x+3) - 1 \)
4. \( y = \log_3(x+3) - 1 \)

The fact that finding the logarithm of a non-positive number (negative or zero) is not possible in the real number system allows us to find the domains of a variety of logarithmic functions.

**Exercise #4:** Determine the domain of the function \( y = \log_2(3x-4) \). State your answer in set-builder notation.

All logarithms with bases larger than 1 are **always increasing**. This increasing nature can be seen by calculating their **average rate of change**.

**Exercise #5:** Consider the common log, or log base 10, \( f(x) = \log(x) \).

(a) Set up and evaluate an expression for the average rate of change of \( f(x) \) over the interval \( 1 \leq x \leq 10 \)

(b) Set up and evaluate an expression for the average rate of change of \( f(x) \) over the interval \( 1 \leq x \leq 100 \).

(c) What do these two answers tell you about the changing slope of this function?
**GRAPHS OF LOGARITHMS**

**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

1. The domain of \( y = \log_3 (x + 5) \) in the real numbers is

   (1) \( \{ x | x > 0 \} \)  
   (3) \( \{ x | x > 5 \} \)

   (2) \( \{ x | x > -5 \} \)  
   (4) \( \{ x | x \geq -4 \} \)

2. Which of the following equations describes the graph shown below?

   (1) \( y = \log_5 x \)  
   (3) \( y = \log_3 x \)

   (2) \( y = \log_2 x \)  
   (4) \( y = \log_4 x \)

3. Which of the following represents the \( y \)-intercept of the function \( y = \log_2 (32 - x) - 1 \)?

   (1) 8  
   (3) \(-1\)

   (2) \(-4\)  
   (4) 4

4. Which of the following values of \( x \) is not in the domain of \( f(x) = \log_5 (10 - 2x) \)?

   (1) \(-3\)  
   (3) 5

   (2) 0  
   (4) 4

5. Which of the following is true about the function \( y = \log_4 (x + 16) - 1 \)?

   (1) It has an \( x \)-intercept of 4 and a \( y \)-intercept of \(-1\).

   (2) It has \( x \)-intercept of \(-12\) and a \( y \)-intercept of 1.

   (3) It has an \( x \)-intercept of \(-16\) and a \( y \)-intercept of 1.

   (4) It has an \( x \)-intercept of \(-16\) and a \( y \)-intercept of \(-1\).
6. Determine the domains of each of the following logarithmic functions. State your answers using any accepted notation. Be sure to show the inequality that you are solving to find the domain and the work you use to solve the inequality.

(a) \( y = \log_3(2x - 1) \) 
(b) \( y = \log(6 - x) \)

7. Graph the logarithmic function \( y = \log_4 x \) on the graph paper given. For a method, see Exercise #1.

**Reasoning**

8. Logarithmic functions whose bases are larger than 1 tend to increase **very slowly** as \( x \) increases. Let's investigate this for \( f(x) = \log_2(x) \).

(a) Find the value of \( f(1), f(2), f(4), \) and \( f(8) \) without your calculator.

(b) For what value of \( x \) will \( \log_2(x) = 10 \)? For what value of \( x \) will \( \log_2(x) = 20 \)?
Logarithms have properties, just as exponents do, that are important to learn because they allow us to solve a variety of problems where logarithms are involved. Keep in mind that since logarithms give exponents, the laws that govern them should be similar to those that govern exponents. Below is a summary of these laws.

**EXPONENT AND LOGARITHM LAWS**

<table>
<thead>
<tr>
<th>LAW</th>
<th>EXPO NENT VERSION</th>
<th>LOGARITHM VERSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product</td>
<td>$b^x \cdot b^y = b^{x+y}$</td>
<td>$\log_b (x \cdot y) = \log_b x + \log_b y$</td>
</tr>
<tr>
<td>Quotient</td>
<td>$\frac{b^x}{b^y} = b^{x-y}$</td>
<td>$\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y$</td>
</tr>
<tr>
<td>Power</td>
<td>$(b^x)^y = b^{xy}$</td>
<td>$\log_b (x^y) = y \cdot \log_b x$</td>
</tr>
</tbody>
</table>

**Exercise #1:** Which of the following is equal to $\log_3 (9x)$?

1. $\log_3 2 + \log_3 x$
2. $2 \log_3 x$
3. $2 + \log_3 x$
4. $x + \log_3 2$

**Exercise #2:** The expression $\log \left( \frac{x^2}{1000} \right)$ can be written in equivalent form as

1. $2 \log x - 3$
2. $\log 2x - 3$
3. $2 \log x - 6$
4. $\log 2x - 6$

**Exercise #3:** If $a = \log 3$ and $b = \log 2$ then which of the following correctly expresses the value of $\log 12$ in terms of $a$ and $b$?

1. $a^2 + b$
2. $a + b^2$
3. $2a + b$
4. $a + 2b$

**Exercise #4:** Which of the following is equivalent to $\log_2 \left( \sqrt[5]{x} \right)$?

1. $\sqrt[5]{\log_2 x - 5 \log_2 y}$
2. $2 \log_2 x + 5 \log_2 y$
3. $\frac{1}{2} \log_2 x - 5 \log_2 y$
4. $2 \log_2 x - 5 \log_2 y$
Exercise #5: The value of $\log_3\left(\frac{\sqrt{5}}{27}\right)$ is equal to

(1) $\frac{\log_3 5 - 6}{2}$          (3) $\frac{\log_3 5 - 3}{2}$  
(2) $2 \log_3 5 + 3$              (4) $2 \log_3 5 - 3$           

Exercise #6: If $f(x) = \log(x)$ and $g(x) = 100x^3$ then $f(g(x)) =$

(1) $100 \log x$            (3) $300 \log x$
(2) $6 + \log x$            (4) $2 + 3 \log x$

Exercise #7: The logarithmic expression $\log_2 \sqrt{32x^7}$ can be rewritten as

(1) $\sqrt{\log_2 35x}$          (3) $\sqrt{5 + 7 \log_2 x}$
(2) $\frac{5 + 7 \log_2 x}{2}$    (4) $\frac{35 + \log_2 x}{2}$

Exercise #8: If $\log 7 = k$ then $\log (4900)$ can be written in terms of $k$ as

(1) $2(k + 1)$        (3) $2(k - 3)$
(2) $2k - 1$          (4) $2k + 1$

The logarithm laws are important for future study in mathematics and science. Being fluent with them is essential. Arguably, the most important of the three laws is the power law. In the next exercise, we will examine it more closely.

Exercise #9: Consider the expression $\log_2 \left(8^x\right)$.

(a) Using the third logarithm law (the Product Law), rewrite this as equivalent product and simplify.

(b) Test the equivalency of these two expressions for $x = 0, 1, \text{ and } 2$.

(c) Show that $\log_2 \left(8^x\right) = 3x$ by rewriting $8$ as $2^3$. 
LOGARITHM LAWS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is not equivalent to \( \log 36 \)?
   (1) \( \log 2 + \log 18 \)  (3) \( \log 30 + \log 6 \)
   (2) \( 2 \log 6 \)  (4) \( \log 4 + \log 9 \)

2. The \( \log_3 20 \) can be written as
   (1) \( 2 \log_3 2 + \log_3 5 \)  (3) \( \log_3 15 + \log_3 5 \)
   (2) \( 2 \log_3 10 \)  (4) \( 2 \log_3 4 + 3 \log_3 4 \)

3. Which of the following is equivalent to \( \log \left( \frac{x^3}{\sqrt[3]{y}} \right) \)?
   (1) \( \log x - \log y \)  (3) \( 3 \log x - \frac{1}{3} \log y \)
   (2) \( 9 \log (x - y) \)  (4) \( \log (3x) - \log \left( \frac{y}{3} \right) \)

4. The difference \( \log_2 (3) - \log_2 (12) \) is equal to
   (1) \( -2 \)  (3) \( \frac{1}{4} \)
   (2) \( -\frac{1}{2} \)  (4) \( 4 \)

5. If \( \log 5 = p \) and \( \log 2 = q \) then \( \log 200 \) can be written in terms of \( p \) and \( q \) as
   (1) \( 4p + q \)  (3) \( 2(p + q) \)
   (2) \( 2p + 3q \)  (4) \( 3p + 2q \)
6. When rounded to the nearest hundredth, \( \log_7 7 = 1.77 \). Which of the following represents the value of \( \log_3 63 \) to the nearest hundredth? Hint: write 63 as a product involving 7.

   (1) 3.54  
   (2) 8.77  
   (3) 3.77  
   (4) 15.93

7. The expression \( 4 \log x - \frac{1}{2} \log y + 3 \log z \) can be rewritten equivalently as

   (1) \( \log \left( \frac{x^4 z^3}{\sqrt{y}} \right) \)  
   (2) \( \log \left( \frac{6xz}{y} \right) \)  
   (3) \( \log \left( \frac{x^4 z^3}{2y} \right) \)  
   (4) \( \log \left( \frac{6x^4 z^3}{y} \right) \)

8. If \( k = \log_2 3 \) then \( \log_2 48 = \)

   (1) \( 2k + 3 \)  
   (2) \( 3k + 1 \)  
   (3) \( k + 8 \)  
   (4) \( k + 4 \)

9. If \( g(x) = 8x^6 \) and \( f(x) = \log_4 (2x) \) then \( f\left(g(x)\right) = ? \)

   (1) \( 4 \log_4 x + 1 \)  
   (2) \( 3 \left( \log_4 x + 2 \right) \)  
   (3) \( 2(3 \log_4 x + 1) \)  
   (4) \( 6 \log_4 x + 4 \)

**REASONING**

10. Consider the exponential equation \( 4^x = 30 \).

   (a) Between what two consecutive integers must the solution to this equation lie? Explain your reasoning.

   (b) Write \( \log \left(4^x\right) \) as an equivalent product using the third logarithm law.

   (c) The solution to the original equation is \( x = \frac{\log(30)}{\log(4)} \), can you see why based on (b)? Evaluate this expression and check to see if it is correct.
SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS
COMMON CORE ALGEBRA II

Earlier in this unit, we used the Method of Common Bases to solve exponential equations. This technique is quite limited, however, because it requires the two sides of the equation to be expressed using the same base. A more general method utilizes our calculators and the third logarithm law:

**The Third Logarithm Law**

\[ \log_b \left( a^x \right) = x \log_b \ a \]

*Exercise #1:* Solve: \(4^x = 8\) using (a) common bases and (b) the logarithm law shown above.

(a) Method of Common Bases \hspace{1cm} (b) Logarithm Approach

The beauty of this logarithm law is that it removes the variable from the exponent. This law, in combination with the logarithm base 10, the common log, allows us to solve almost any exponential equation using calculator technology.

*Exercise #2:* Solve each of the following equations for the value of \(x\). Round your answers to the nearest hundredth.

(a) \(5^x = 18\) \hspace{1cm} (b) \(4^x = 100\) \hspace{1cm} (c) \(2^x = 1560\)

These equations can become more complicated, but each and every time we will use the logarithm law to transform an exponential equation into one that is more familiar (linear only for now).

*Exercise #3:* Solve each of the following equations for \(x\). Round your answers to the nearest hundredth.

(a) \(6^{x+3} = 50\) \hspace{1cm} (b) \((1.03)^{x-5} = 2\)
Now that we are familiar with this method, we can revisit some of our exponential models from earlier in the unit. Recall that for an exponential function that is growing:

If quantity $Q$ is known to increase by a fixed percentage $p$, in decimal form, then $Q$ can be modeled by

$$Q(t) = Q_0 (1 + p)^t$$

where $Q_0$ represents the amount of $Q$ present at $t = 0$ and $t$ represents time.

**Exercise #4:** A biologist is modeling the population of bats on a tropical island. When he first starts observing them, there are 104 bats. The biologist believes that the bat population is growing at a rate of 3% per year.

(a) Write an equation for the number of bats, $B(t)$, as a function of the number of years, $t$, since the biologist started observing them.  

(b) Using your equation from part (a), algebraically determine the number of years it will take for the bat population to reach 200. Round your answer to the nearest year.

**Exercise #5:** A stock has been declining in price at a steady pace of 5% per week. If the stock started at a price of $22.50 per share, determine algebraically the number of weeks it will take for the price to reach $10.00. Round your answer to the nearest week.

As a final discussion, we return to evaluating logarithms using our calculator. Many modern calculators can find a logarithm of any base. Some still only have the common log (base 10) and another that we will soon see. But, we can still express our answers in terms of logarithms.

**Exercise #6:** Find the solution to each of the following exponential equations in terms of a logarithm with the same base as the exponential equation.

(a) $4(2)^x - 3 = 17$  

(b) $17(5)^{\frac{x}{3}} = 4$
SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following values, to the nearest hundredth, solves: \(7^x = 500\)?
   (1) 3.19  (2) 3.83  (3) 2.74  (4) 2.17

2. The solution to \(2^{\sqrt{x}} = 52\), to the nearest tenth, is which of the following?
   (1) 7.3  (2) 9.1  (3) 11.4  (4) 17.1

3. To the nearest hundredth, the value of \(x\) that solves \(5^{x-4} = 275\) is
   (1) 6.73  (2) 5.74  (3) 8.17  (4) 7.49

4. Solve each of the following exponential equations. Round each of your answers to the nearest hundredth.
   (a) \(9^{x-3} = 250\)
   (b) \(50(2)^x = 1000\)
   (c) \(5^{\sqrt{x}} = 35\)

5. Solve each of the following exponential equations. Be careful with your use of parentheses. Express each answer to the nearest hundredth.
   (a) \(6^{2x-5} = 300\)
   (b) \(\left(\frac{1}{2}\right)^{\frac{x}{3}} = \frac{1}{6}\)
   (c) \(500(1.02)^{\frac{x}{12}} = 2300\)
APPLICATIONS

6. The population of Red Hook is growing at a rate of 3.5% per year. If its current population is 12,500, in how many years will the population exceed 20,000? Round your answer to the nearest year. Only an algebraic solution is acceptable.

7. A radioactive substance is decaying such that 2% of its mass is lost every year. Originally there were 50 kilograms of the substance present.
   (a) Write an equation for the amount, $A$, of the substance left after $t$-years.
   (b) Find the amount of time that it takes for only half of the initial amount to remain. Round your answer to the nearest tenth of a year.

REASONING

8. If a population doubles every 5 years, how many years will it take for the population to increase by 10 times its original amount?

   First: If the population gets multiplied by 2 every 5 years, what does it get multiplied by each year? Use this to help you answer the question.

9. Find the solution to the general exponential equation $a(b)^{ct} = d$, in terms of the constants $a$, $c$, $d$ and the logarithm of base $b$. Think about reversing the order of operations in order to solve for $x$. 
THE NUMBER e AND THE NATURAL LOGARITHM
COMMON CORE ALGEBRA II

There are many numbers in mathematics that are more important than others because they find so many uses in either mathematics or science. Good examples of important numbers are 0, 1, \( i \), and \( \pi \). In this lesson you will be introduced to an important number given the letter \( e \) for its “inventor” Leonhard Euler (1707-1783). This number plays a crucial role in Calculus and more generally in modeling exponential phenomena.

### THE NUMBER e

1. Like \( \pi \), \( e \) is irrational.
2. \( e \approx 2.72 \)
3. Used in Exponential Modeling

**Exercise #1:** Which of the graphs below shows \( y = e^x \)? Explain your choice. Check on your calculator.

(1) ![Graph 1](image1)
(2) ![Graph 2](image2)
(3) ![Graph 3](image3)
(4) ![Graph 4](image4)

**Explanation:**

Very often \( e \) is involved in exponential modeling of both increasing and decreasing quantities. The creation of these models is beyond the scope of this course, but we can still work with them.

**Exercise #2:** A population of llamas on a tropical island can be modeled by the equation \( P = 500e^{0.035t} \), where \( t \) represents the number of years since the llamas were first introduced to the island.

(a) How many llamas were initially introduced at \( t = 0 \)? Show the calculation that leads to your answer.

(b) Algebraically determine the number of years for the population to reach 600. Round your answer to the nearest tenth of a year.
Because of the importance of \( y = e^x \), its inverse, known as the **natural logarithm**, is also important.

### The Natural Logarithm

The inverse of \( y = e^x \): \( y = \ln x \) \((y = \log_e x)\)

The natural logarithm, like all logarithms, gives an exponent as its output. In fact, it gives the power that we must raise \( e \) to in order to get the input.

**Exercise #3:** *Without* the use of your calculator, determine the values of each of the following.

(a) \( \ln(e) \)  
(b) \( \ln(1) \)  
(c) \( \ln(e^5) \)  
(d) \( \ln(\sqrt{e}) \)

The natural logarithm follows the three basic logarithm laws that all logarithms follow. The following problems give additional practice with these laws.

**Exercise #4:** Which of the following is equivalent to \( \ln\left(\frac{x^3}{e^2}\right) \)?

1. \( \ln x + 6 \)  
2. \( 3 \ln x - 2 \)  
3. \( 3 \ln x - 6 \)  
4. \( \ln x - 9 \)

**Exercise #5:** A hot liquid is cooling in a room whose temperature is constant. Its temperature can be modeled using the exponential function shown below. The temperature, \( T \), is in degrees Fahrenheit and is a function of the number of minutes, \( m \), it has been cooling.

\[
T(m) = 101e^{-0.03m} + 67
\]

(a) What was the initial temperature of the water at \( m = 0 \). Do without using your calculator.  
(b) How do you interpret the statement that \( T(60) = 83.7 \)?

(c) Using the natural logarithm, determine algebraically when the temperature of the liquid will reach 100 °F. Show the steps in your solution. Round to the nearest tenth of a minute.  
(d) On average, how many degrees are lost per minute over the interval \( 10 \leq m \leq 30 \)? Round to the nearest tenth of a degree.
THE NUMBER e AND THE NATURAL LOGARITHM
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is closest to the $y$-intercept of the function whose equation is $y = 10e^{x+1}$?
   (1) 10       (3) 27
   (2) 18       (4) 52

2. On the grid below, the solid curve represents $y = e^x$. Which of the following exponential functions could describe the dashed curve? Explain your choice.
   (1) $y = \left(\frac{1}{2}\right)^x$   (3) $y = 2^x$
   (2) $y = e^{-x}$                (4) $y = 4^x$

3. The logarithmic expression $\ln\left(\frac{\sqrt{e}}{y^3}\right)$ can be rewritten as
   (1) $3\ln y - 2$       (3) $\frac{\ln y - 6}{2}$
   (2) $\frac{1 - 6\ln y}{2}$   (4) $\sqrt{\ln y} - 3$

4. Which of the following values of $t$ solves the equation $5e^{2t} = 15$?
   (1) $\ln\frac{15}{10}$         (3) $2\ln 3$
   (2) $\frac{1}{2\ln 5}$         (4) $\frac{\ln 3}{2}$

5. At which of the following values of $x$ does $f(x) = 2e^{2x} - 32$ have a zero?
   (1) $\ln\frac{5}{2}$         (3) $\ln 8$
   (2) $\ln 4$                  (4) $y = \ln\frac{2}{5}$
6. For the equation \( ae^{ct} = d \), solve for the variable \( t \) in terms of \( a, c, \) and \( d \). Express your answer in terms of the natural logarithm.

APPLICATIONS

7. Flu is spreading exponentially at a school. The number of new flu patients can be modeled using the equation \( F = 10e^{0.12d} \), where \( d \) represents the number of days since 10 students had the flu.

(a) How many days will it take for the number of new flu patients to equal 50? Determine your answer algebraically using the natural logarithm. Round your answer to the nearest day.

(b) Find the average rate of change of \( F \) over the first three weeks, i.e. \( 0 \leq d \leq 21 \). Show the calculation that leads to your answer. Give proper units and round your answer to the nearest tenth. What is the physical interpretation of your answer?

8. The savings in a bank account can be modeled using \( S = 1250e^{0.045t} \), where \( t \) is the number of years the money has been in the account. Determine, to the nearest tenth of a year, how long it will take for the amount of savings to double from the initial amount deposited of $1250.
In the worlds of investment and debt, interest is added onto a principal in what is known as compound interest. The percent rate is typically given on a yearly basis, but could be applied more than once a year. This is known as the compounding frequency. Let’s take a look at a typical problem to understand how the compounding frequency changes how interest is applied.

**Exercise #1:** A person invests $500 in an account that earns a nominal yearly interest rate of 4%.

(a) How much would this investment be worth in 10 years if the compounding frequency was once per year? Show the calculation you use.

(b) If, on the other hand, the interest was applied four times per year (known as quarterly compounding), why would it not make sense to multiply by 1.04 each quarter?

(c) If you were told that an investment earned 4% per year, how much would you assume was earned per quarter? Why?

(d) Using your answer from part (c), calculate how much the investment would be worth after 10 years of quarterly compounding? Show your calculation.

So, the pattern is fairly straightforward. For a shorter compounding period, we get to apply the interest more often, but at a lower rate.

**Exercise #2:** How much would $1000 invested at a nominal 2% yearly rate, compounded monthly, be worth in 20 years? Show the calculations that lead to your answer.

(1) $1485.95  
(2) $1491.33  
(3) $1033.87  
(4) $1045.32

This pattern is formalized in a classic formula from economics that we will look at in the next exercise.

**Exercise #3:** For an investment with the following parameters, write a formula for the amount the investment is worth, $A$, after $t$-years.

\[ P = \text{amount initially invested} \]
\[ r = \text{nominal yearly rate} \]
\[ n = \text{number of compounds per year} \]

\[ A(t) = \]
The rate in Exercise #1 was referred to as nominal (in name only). It's known as this, because you effectively earn more than this rate if the compounding period is more than once per year. Because of this, bankers refer to the effective rate, or the rate you would receive if compounded just once per year. Let's investigate this.

**Exercise #4:** An investment with a nominal rate of 5% is compounded at different frequencies. Give the effective yearly rate, accurate to two decimal places, for each of the following compounding frequencies. Show your calculation.

(a) Quarterly  
(b) Monthly  
(c) Daily

We could compound at smaller and smaller frequency intervals, eventually compounding all moments of time. In our formula from Exercise #3, we would be letting \( n \) approach infinity. Interestingly enough, this gives rise to continuous compounding and the use of the natural base \( e \) in the famous **continuous compound interest** formula.

### CONTINUOUS COMPOUND INTEREST

For an initial principal, \( P \), compounded continuously at a nominal yearly rate of \( r \), the investment would be worth an amount \( A \) given by:

\[
A(t) = Pe^{rt}
\]

**Exercise #5:** A person invests $350 in a bank account that promises a nominal rate of 2% continuously compounded.

(a) Write an equation for the amount this investment would be worth after \( t \)-years.  
(b) How much would the investment be worth after 20 years?  
(c) Algebraically determine the time it will take for the investment to reach $400. Round to the nearest tenth of a year.  
(d) What is the effective annual rate for this investment? Round to the nearest hundredth of a percent.
APPLICATIONS

1. The value of an initial investment of $400 at 3% nominal interest compounded quarterly can be modeled using which of the following equations, where \( t \) is the number of years since the investment was made?

   (1) \( A = 400 \left(1.0075\right)^{4t} \)
   (2) \( A = 400 \left(1.0075\right)^t \)
   (3) \( A = 400 \left(1.03\right)^{4t} \)
   (4) \( A = 400 \left(1.0303\right)^{4t} \)

2. Which of the following represents the value of an investment with a principal of $1500 with a nominal interest rate of 2.5% compounded monthly after 5 years?

   (1) $1,697.11  
   (2) $1,699.50  
   (3) $4,178.22  
   (4) $5,168.71

3. Franco invests $4,500 in an account that earns a 3.8% nominal interest rate compounded continuously. If he withdraws the profit from the investment after 5 years, how much has he earned on his investment?

   (1) $858.92  
   (2) $912.59  
   (3) $922.50  
   (4) $941.62

4. An investment that returns a nominal 4.2% yearly rate, but is compounded quarterly, has an effective yearly rate closest to

   (1) 4.21%  
   (2) 4.24%  
   (3) 4.27%  
   (4) 4.32%

5. If an investment's value can be modeled with \( A = 325 \left(1 + \frac{0.27}{12}\right)^{12t} \) then which of the following describes the investment?

   (1) The investment has a nominal rate of 27% compounded every 12 years.
   (2) The investment has a nominal rate of 2.7% compounded every 12 years.
   (3) The investment has a nominal rate of 27% compounded 12 times per year.
   (4) The investment has a nominal rate of 2.7% compounded 12 times per year.
6. An investment of $500 is made at 2.8% nominal interest compounded quarterly.

(a) Write an equation that models the amount \( A \) \( t \)-years after the principal has been invested.

(b) How much is the investment worth after 10 years?

(c) Algebraically determine the number of years it will take for the investment to reach a worth of $800. Round to the nearest hundredth.

(d) Why does it make more sense to round your answer in (c) to the nearest quarter? State the final answer rounded to the nearest quarter.

REASONING

7. The formula \( A = P \left(1 + \frac{r}{n}\right)^{nt} \) can be rearranged using properties of exponents as \( A = P \left(1 + \frac{r}{n}\right)^{nt} \). Explain what the term \( \left(1 + \frac{r}{n}\right)^n \) helps to calculate.

8. The formula \( A = Pe^{rt} \) calculates the amount an investment earning a nominal rate of \( r \) compounded continuously is worth. Show that the amount of time it takes for the investment to double in value is given by the expression \( \frac{\ln 2}{r} \).
Newton's Law of Cooling
Common Core Algebra II

The temperature of a cooling liquid in a large room with a steady temperature is a great example of a type of a transformed exponential function. We will explore this today to see how a simple exponential function can be used to build a more complex one.

Exercise #1: Consider the decreasing exponential function \( f(x) = 8 \left( \frac{1}{2} \right)^x \).

(a) Use your calculator to sketch the graph using the window indicated.

(b) Clearly the value of \( y \) gets smaller as \( x \) gets larger. Does it ever reach zero? Why or why not?

Exercise #2: Assume a liquid starts at a temperature of 200 °F and begins to cool in a room that is at a steady temperature of 70 °F.

(a) Draw a rough sketch of what you believe the liquid's temperature function looks like as time increases.

(b) Based on your graph from (a) and your work in Exercise #1, why would an equation of the form \( y = a(b)^t \) not model this cooling well (assuming that \( 0 < b < 1 \))? How could we modify this equation to make it model the situation more realistically?
**Exercise #3:** Let's stick with the same cooling liquid that we had before, i.e. one that starts at 200 °F and cools down in a room that is held at a constant temperature of 70 °F. We will now model this cooling Fahrenheit temperature using the equation $F(t) = a(b)^t + c$, where $a$, $b$, and $c$ are all parameters (constants) in the model and $t$ is time in minutes.

(a) Which of these constants is equal to 70 and why? Think about the last problem.

(b) None of these constants is equal to 200, but $T(0) = 200$. What constant does this let you solve for? Find its value.

(c) In order to find the value of $b$ what additional information would we need?

(d) Determine the value of $b$ if the temperature, after 5 minutes, is 153 °F. Round to the nearest hundredth.

(e) What is the temperature of the liquid after half an hour? An hour? Two hours?

(f) Using your calculator, sketch the graph of the liquid's temperature. Decide on an appropriate window and label it on your axes.

(g) Algebraically determine, to the nearest tenth of a minute, when the temperature reaches 100 °F.
**Newton's Law of Cooling**

**COMMON CORE ALGEBRA II HOMEWORK**

**Fluency**

1. For the function \( f(x) = a(b)^x \), where \( 0 < b < 1 \), what value does the function's output approach as \( x \) gets very large?

2. For the function \( g(x) = a(b)^x + k \), where \( 0 < b < 1 \), what value does the function's output approach as \( x \) gets very large?

3. Given \( f(t) = a(b)^t + c \), which of the following represents the \( y \)-intercept of this function? Show how you arrived at your choice.

   (1) \( a \)  
   (2) \( a + c \)  
   (3) \( c \)  
   (4) \( b + c \)

**Applications**

4. A liquid starts at an initial temperature of 175 °C and cools down in a room held at a constant temperature of 16 °C. Its temperature can be modeled, as a function of time cooling, by the equation \( y = a(b)^x + c \). Which of the following statements is true?

   (1) \( a = 159 \) and \( c = 16 \)  
   (2) \( a = 16 \) and \( c = 159 \)  
   (3) \( a = 175 \) and \( c = 16 \)  
   (4) \( a = 16 \) and \( c = 175 \)

5. A cooling liquid has a temperature given by the function \( T(m) = 132(.83)^m + 40 \), where \( m \) is the number of minutes it has been cooling. Which of the following temperatures did the liquid start at?

   (1) 40  
   (2) 92  
   (3) 172  
   (4) 132
6. A liquid starts at a temperature of 190 °F and cools down in a room held at a constant 65 °F. After 10 minutes of cooling, it is at a temperature of 92 °F. The Fahrenheit temperature, \( F \), can be modeled as a function of time in minutes, \( t \), by the equation:

\[
F(t) = a(b)^t + c
\]

(a) Determine the values of the parameters \( a \), \( b \), and \( c \). Round the value of \( b \) to the nearest hundredth. State the equation of your final model. Show the work that leads to each of your answers.

(b) Algebraically, determine the number of minutes it will take for the temperature to reach 70 °F. Round to the nearest tenth of a minute.

REASONING

7. When we model the temperature of a cooling liquid using the equation \( T = a(b)^x + c \), we have learned that the value of \( c \) represents the steady temperature of the room. The quantity \( a(b)^x \) does model something physically. Can you determine what it is?

8. A Warming Liquid - A liquid is taken out of a refrigerator and placed in a warmer room, where its temperature, in F, increases over time. It can be modeled using the equation \( T(m) = 74 - 39(0.87)^m \).

(a) What temperature did the liquid start at? Show the work that leads to your answer. (b) What is the temperature of the room?
UNIT #5

SEQUENCES AND SERIES

Lesson #1 – Sequences
Lesson #2 – Arithmetic and Geometric Sequences
Lesson #3 – Summation Notation
Lesson #4 – Arithmetic Series
Lesson #5 – Geometric Series
Lesson #6 – Mortgage Payments
SEQUENCES
COMMON CORE ALGEBRA II

In Common Core Algebra I, you studied sequences, which are ordered lists of numbers. Sequences are extremely important in mathematics, both theoretical and applied. A sequence is formally defined as a function that has as its domain the set of positive integers, i.e. \{1, 2, 3, ..., n\}.

**Exercise #1:** A sequence is defined by the equation \(a(n) = 2n - 1\).

(a) Find the first three terms of this sequence, denoted by \(a_1, a_2,\) and \(a_3\).

(b) Which term has a value of 53?

(c) Explain why there will not be a term that has a value of 70.

Recall that sequences can also be described by using recursive definitions. When a sequence is defined recursively, terms are found by operations on previous terms.

**Exercise #2:** A sequence is defined by the recursive formula: \(f(n) = f(n-1) + 5\) with \(f(1) = -2\).

(a) Generate the first five terms of this sequence. Label each term with proper function notation.

(b) Determine the value of \(f(20)\). Hint – think about how many times you have added 5 to -2.

**Exercise #3:** Determine a recursive definition, in terms of \(f(n)\), for the sequence shown below. Be sure to include a starting value.

5, 10, 20, 40, 80, 160, ...

**Exercise #4:** For the recursively defined sequence \(t_n = (t_{n-1})^2 + 2\) and \(t_1 = 2\), the value of \(t_4\) is

- (1) 18
- (2) 38
- (3) 456
- (4) 1446

COMMON CORE ALGEBRA II, UNIT #5 – SEQUENCES AND SERIES – LESSON #1
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Exercise #5: One of the most well-known sequences is the Fibonacci, which is defined recursively using two previous terms. Its definition is given below.

\[ f(n) = f(n-1) + f(n-2) \text{ and } f(1) = 1 \text{ and } f(2) = 1 \]

Generate values for \( f(3), f(4), f(5), \) and \( f(6) \) (in other words, then next four terms of this sequence).

It is often possible to find algebraic formulas for simple sequences, and this skill should be practiced.

Exercise #6: Find an algebraic formula \( a(n) \), similar to that in Exercise #1, for each of the following sequences. Recall that the domain that you map from will be the set \( \{1, 2, 3, ..., n\} \).

(a) 4, 5, 6, 7, ...  
(b) 2, 4, 8, 16, ...  
(c) 5, 5, 5, 5, ...  
(d) -1, 1, -1, 1, ...  
(e) 10, 15, 20, 25, ...  
(f) 1, 1, 1, 1, ...  

Exercise #7: Which of the following would represent the graph of the sequence \( a_n = 2n + 1 \)? Explain your choice.

(1)  
(2)  
(3)  
(4)  

Explanation:
SEQUENCES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Given each of the following sequences defined by formulas, determine and label the first four terms. A variety of different notations is used below for practice purposes.

   (a) \( f(n) = 7n + 2 \)  
   (b) \( a_n = n^2 - 5 \)  
   (c) \( t(n) = \left( \frac{2}{3} \right)^n \)  
   (d) \( t_n = \frac{1}{n+1} \)

2. Sequences below are defined recursively. Determine and label the next three terms of the sequence.

   (a) \( f(1) = 4 \) and \( f(n) = f(n-1) + 8 \)  
   (b) \( a(n) = a(n-1) \cdot \frac{1}{2} \) and \( a(1) = 24 \)

   (c) \( b_n = b_{n-1} + 2n \) with \( b_1 = 5 \)  
   (d) \( f(n) = 2f(n-1) - n^2 \) and \( f(1) = 4 \)

3. Given the sequence 7, 11, 15, 19, ..., which of the following represents a formula that will generate it?

   (1) \( a(n) = 4n + 7 \)  
   (2) \( a(n) = 3n + 4 \)  
   (3) \( a(n) = 3n + 7 \)  
   (4) \( a(n) = 4n + 3 \)

4. A recursive sequence is defined by \( a_{n+1} = 2a_n - a_{n-1} \) with \( a_1 = 0 \) and \( a_2 = 1 \). Which of the following represents the value of \( a_5 \)?

   (1) 8  
   (2) -7  
   (3) 3  
   (4) 4

5. Which of the following formulas would represent the sequence 10, 20, 40, 80, 160, ...?

   (1) \( a_n = 10^n \)  
   (2) \( a_n = 10(2)^n \)  
   (3) \( a_n = 5(2)^n \)  
   (4) \( a_n = 2n + 10 \)
6. For each of the following sequences, determine an algebraic formula, similar to Exercise #4, that defines the sequence.

(a) 5, 10, 15, 20, …  
(b) 3, 9, 27, 81, …  
(c) \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \)

7. For each of the following sequences, state a recursive definition. Be sure to include a starting value or values.

(a) 8, 6, 4, 2, …  
(b) 2, 6, 18, 54, …  
(c) 2, –2, 2, –2, …

APPLICATIONS

8. A tiling pattern is created from a single square and then expanded as shown. If the number of squares in each pattern defines a sequence, then determine the number of squares in the seventh pattern. Explain how you arrived at your choice. Can you write a recursive definition for the pattern?

![Tiling Pattern Image]

REASONING

9. Consider a sequence defined similarly to the Fibonacci, but with a slight twist:

\[ f(n) = f(n-1) - f(n-2) \text{ with } f(1) = 2 \text{ and } f(2) = 5 \]

Generate terms \( f(3), f(4), f(5), f(6), f(7), \) and \( f(8) \). Then, determine the value of \( f(25) \).
ARITHMETIC AND GEOMETRIC SEQUENCES
COMMON CORE ALGEBRA II

In Common Core Algebra I, you studied two particular sequences known as arithmetic (based on constant addition to get the next term) and geometric (based on constant multiplying to get the next term). In this lesson, we will review the basics of these two sequences.

### ARITHMETIC SEQUENCE RECURSIVE DEFINITION

Given \( f(1) \), then \( f(n) = f(n-1) + d \) or given \( a_1 \) then \( a_n = a_{n-1} + d \)

where \( d \) is called the **common difference** and can be positive or negative.

**Exercise #1:** Generate the next three terms of the given arithmetic sequences.

(a) \( f(n) = f(n-1) + 6 \) with \( f(1) = 2 \)

(b) \( a_n = a_{n-1} + \frac{1}{2} \) and \( a_1 = \frac{3}{2} \)

**Exercise #2:** For some number \( t \), the first three terms of an arithmetic sequence are \( 2t \), \( 5t-1 \), and \( 6t+2 \). What is the numerical value of the fourth term? Hint: first set up an equation that will solve for \( t \).

It is important to be able to determine a general term of an arithmetic sequence based on the value of the index variable (the subscript). The next exercise walks you through the thinking process involved.

**Exercise #3:** Consider \( a_n = a_{n-1} + 3 \) with \( a_1 = 5 \).

(a) Determine the value of \( a_2 \), \( a_3 \), and \( a_4 \).

(b) How many times was 3 added to 5 in order to produce \( a_4 \)?

(c) Use your result from part (b) to quickly find the value of \( a_{50} \).

(d) Write a formula for the \( n^{th} \) term of an arithmetic sequence, \( a_n \), based on the first term, \( a_1 \), \( d \) and \( n \).
Exercise #4: Given that \( a_1 = 6 \) and \( a_4 = 18 \) are members of an arithmetic sequence, determine the value of \( a_{20} \).

Geometric sequences are defined very similarly to arithmetic, but with a multiplicative constant instead of an additive one.

**GEOMETRIC SEQUENCE RECURSIVE DEFINITION**

Given \( f(1) \) then \( f(n) = f(n-1) \cdot r \) or given \( a_1 \), then \( a_n = a_{n-1} \cdot r \)

where \( r \) is called the **common ratio** and can be positive or negative and is often fractional.

Exercise #5: Generate the next three terms of the geometric sequences given below.

(a) \( a_1 = 4 \) and \( r = 2 \)
(b) \( f(n) = f(n-1) \cdot \frac{1}{3} \) with \( f(1) = 9 \)
(c) \( t_n = t_{n-1} \cdot \sqrt{2} \) with \( t_1 = 3\sqrt{2} \)

And, like arithmetic, we also need to be able to determine any given term of a geometric sequence based on the first value, the common ratio, and the index.

Exercise #6: Consider \( a_1 = 2 \) and \( a_n = a_{n-1} \cdot 3 \).

(a) Generate the value of \( a_4 \).  
(b) How many times did you need to multiply 2 by 3 in order to find \( a_4 \).  

(c) Determine the value of \( a_{10} \).  
(d) Write a formula for the \( n \)th term of a geometric sequence, \( a_n \), based on the first term, \( a_1 \), \( r \) and \( n \).
ARITHMETIC AND GEOMETRIC SEQUENCES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Generate the next three terms of each arithmetic sequence shown below.
   
   (a) \( a_1 = -2 \) and \( d = 4 \)  
   (b) \( f(n) = f(n-1) - 8 \) with \( f(1) = 10 \)  
   (c) \( a_1 = 3, \ a_2 = 1 \)

2. In an arithmetic sequence \( t_n = t_{n-1} + 7 \). If \( t_1 = -5 \) determine the values of \( t_4 \) and \( t_{20} \). Show the calculations that lead to your answers.

3. If \( x + 4, 2x + 5, \) and \( 4x + 3 \) represent the first three terms of an arithmetic sequence, then find the value of \( x \). What is the fourth term?

4. If \( f(1) = 12 \) and \( f(n) = f(n-1) - 4 \) then which of the following represents the value of \( f(40) \)?
   
   (1) \(-148\)  
   (2) \(-140\)  
   (3) \(-144\)  
   (4) \(-172\)

5. In an arithmetic sequence of numbers \( a_1 = -4 \) and \( a_6 = 46 \). Which of the following is the value of \( a_{12} \)?
   
   (1) \(120\)  
   (2) \(146\)  
   (3) \(92\)  
   (4) \(106\)

6. The first term of an arithmetic sequence whose common difference is 7 and whose 22\(^{\text{nd}}\) term is given by \( a_{22} = 143 \) is which of the following?
   
   (1) \(-25\)  
   (2) \(-4\)  
   (3) \(7\)  
   (4) \(28\)
7. Generate the next three terms of each geometric sequence defined below.

   (a) $a_1 = -8$ with $r = -1$  
   (b) $a_n = a_{n-1} \cdot \frac{3}{2}$ and $a_1 = 16$  
   (c) $f(n) = f(n-1) \cdot -2$ and $f(1) = 5$

8. Given that $a_1 = 5$ and $a_2 = 15$ are the first two terms of a geometric sequence, determine the values of $a_3$ and $a_{10}$. Show the calculations that lead to your answers.

9. In a geometric sequence, it is known that $a_1 = -1$ and $a_4 = 64$. The value of $a_{10}$ is

   (1) $-65,536$  
   (2) $262,144$  
   (3) $512$  
   (4) $-4096$

APPLICATIONS

10. The Koch Snowflake is a mathematical shape known as a fractal that has many fascinating properties. It is created by repeatedly forming equilateral triangles off of the sides of other equilateral triangles. Its first six iterations are shown to the right. The perimeters of each of the figures form a geometric sequence.

   (a) If the perimeter of the first snowflake (the equilateral triangle) is 3, what is the perimeter of the second snowflake? Note: the dashed lines in the second snowflake are not to be counted towards the perimeter. They are only there to show how the snowflake was constructed.

   (b) Given that the perimeters form a geometric sequence, what is the perimeter of the sixth snowflake? Express your answer to the nearest tenth.

   (c) If the this process was allowed to continue forever, explain why the perimeter would become infinitely large.
SUMMATION NOTATION
COMMON CORE ALGEBRA II

Much of our work in this unit will concern **adding the terms of a sequence**. In order to specify this addition or summarize it, we introduce a new notation, known as **summation or sigma notation** that will represent these sums. This notation will also be used later in the course when we want to write formulas used in statistics.

**SUMMATION (SIGMA) NOTATION**

\[ \sum_{i=a}^{n} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(n) \]

where *i* is called the **index variable**, which starts at a value of *a*, ends at a value of *n*, and moves by unit increments (increase by 1 each time).

**Exercise #1:** Evaluate each of the following sums.

(a) \[ \sum_{i=3}^{5} 2i \]

(b) \[ \sum_{k=-1}^{3} k^2 \]

(c) \[ \sum_{j=-2}^{2} 2^j \]

(d) \[ \sum_{i=1}^{5} (-1)^i \]

(e) \[ \sum_{k=0}^{2} (2k + 1) \]

(f) \[ \sum_{i=1}^{3} i(i+1) \]

**Exercise #2:** Which of represents the value of \[ \sum_{i=1}^{4} \frac{1}{i} \]?

(1) \[ \frac{1}{10} \]

(2) \[ \frac{9}{4} \]

(3) \[ \frac{25}{12} \]

(4) \[ \frac{31}{24} \]
Exercise #3: Consider the sequence defined recursively by \( a_n = a_{n-1} + 2a_{n-2} \) and \( a_1 = 0 \) and \( a_2 = 1 \). Find the value of \( \sum_{i=4}^{7} a_i \).

It is also good to be able to place sums into sigma notation. These answers, though, will not be unique.

Exercise #4: Express each sum using sigma notation. Use \( i \) as your index variable. First, consider any patterns you notice amongst the terms involved in the sum. Then, work to put these patterns into a formula and sum.

(a) \( 9 + 16 + 25 + \cdots + 100 \) (b) \( -6 + -3 + 0 + 3 + \cdots + 15 \) (c) \( \frac{1}{25} + \frac{1}{5} + 1 + 5 + \cdots + 625 \)

Exercise #5: Which of the following represents the sum \( 3 + 6 + 12 + 24 + 48 \)?

(1) \( \sum_{i=1}^{5} 3^i \) (2) \( \sum_{i=0}^{4} 3(2)^i \) (3) \( \sum_{i=0}^{4} 6^{i-1} \) (4) \( \sum_{i=3}^{48} i \)

Exercise #6: Some sums are more interesting than others. Determine the value of \( \sum_{i=1}^{99} \left( \frac{1}{i} - \frac{1}{i+1} \right) \). Show your reasoning. This is known as a **telescoping series (or sum)**.
SUMMATION NOTATION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Evaluate each of the following. Place any non-integer answer in simplest rational form.

(a) \( \sum_{i=2}^{5} 4i \)

(b) \( \sum_{k=0}^{3} (k^2 + 1) \)

(c) \( \sum_{j=-2}^{0} (2j + 1) \)

(d) \( \sum_{j=-1}^{3} 2^j \)

(e) \( \sum_{k=0}^{2} (-1)^{2k+1} \)

(f) \( \sum_{i=1}^{3} \log(10^i) \)

(g) \( \sum_{n=1}^{4} \frac{n}{n+1} \)

(h) \( \sum_{j=2}^{4} \frac{(i+1)^2}{4} \)

(i) \( \sum_{k=0}^{3} 256 \cdot 2^k \)

2. Which of the following is the value of \( \sum_{k=0}^{4} (4k + 1) \)?

(1) 53

(2) 45

(3) 37

(4) 80

3. The sum \( \sum_{i=4}^{7} 2^{i-7} \) is equal to

(1) \( \frac{15}{8} \)

(2) \( \frac{3}{2} \)

(3) \( \frac{3}{4} \)

(4) \( \frac{7}{8} \)
4. Write each of the following sums using sigma notation. Use \( k \) as your index variable. Note, there are many correct ways to write each sum (and even more incorrect ways).

(a) \(-2 + 4 + 8 + \cdots + 64 + 128\) \quad (b) \(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{81} + \frac{1}{100}\) \quad (c) \(4 + 9 + 14 + \cdots + 44 + 49\)

5. Which of the following represents the sum \(2 + 5 + 10 + \cdots + 82 + 101\)?

(1) \(\sum_{j=1}^{6} (4j - 3)\) \quad (2) \(\sum_{j=3}^{10} (j - 2)\) \quad (3) \(\sum_{j=1}^{10} (j^2 + 1)\) \quad (4) \(\sum_{j=0}^{11} (4j + 1)\)

6. A sequence is defined recursively by the formula \(b_n = 4b_{n-1} - 2b_{n-2}\) with \(b_1 = 1\) and \(b_2 = 3\). What is the value of \(\sum_{i=3}^{5} b_i\) ? Show the work that leads to your answer.

**REASONING**

6. A curious pattern occurs when we look at the behavior of the sum \(\sum_{k=1}^{n} (2k - 1)\).

(a) Find the value of this sum for a variety of values of \(n\) below:

\[
\begin{align*}
n = 2: \sum_{k=1}^{2} (2k - 1) &= \\
n = 4: \sum_{k=1}^{4} (2k - 1) &=
\end{align*}
\]

\[
\begin{align*}
n = 3: \sum_{k=1}^{3} (2k - 1) &= \\
n = 5: \sum_{k=1}^{5} (2k - 1) &=
\end{align*}
\]

(b) What types of numbers are you summing? \quad (c) Find the value of \(n\) such that \(\sum_{k=1}^{n} (2k - 1) = 196\).

What types of numbers are the sums?
A series is simply the sum of the terms of a sequence. The fundamental definition/notion of a series is below.

**THE DEFINITION OF A SERIES**

If the set \( \{a_1, a_2, a_3, \ldots \} \) represent the elements of a sequence then the series, \( S_n \), is defined by:

\[
S_n = \sum_{i=1}^{n} a_i
\]

In truth, you have already worked extensively with series in previous lessons almost anytime you evaluated a summation problem.

**Exercise #1:** Given the arithmetic sequence defined by \( a_i = -2 \) and \( a_n = a_{n-1} + 5 \), then which of the following is the value of \( S_5 = \sum_{i=1}^{5} a_i \)?

- (1) 32
- (2) 40
- (3) 25
- (4) 27

The sums associated with arithmetic sequences, known as **arithmetic series**, have interesting properties, many applications and values that can be predicted with what is commonly known as **rainbow addition**.

**Exercise #2:** Consider the arithmetic sequence defined by \( a_i = 3 \) and \( a_n = a_{n-1} + 2 \). The series, based on the first eight terms of this sequence, is shown below. Terms have been paired off as shown.

(a) What does each of the paired off sums equal?

(b) Why does it make sense that this sum is constant?

(c) How many of these pairs are there?

(d) Using your answers to (a) and (c) find the value of the sum using a multiplicative process.

(e) Generalize this now and create a formula for an arithmetic series sum based only on its first term, \( a_1 \), its last term, \( a_n \), and the number of terms, \( n \).
**SUM OF AN ARITHMETIC SERIES**

Given an arithmetic series with \( n \) terms, \( \{a_1, a_2, \ldots, a_n\} \), then its sum is given by:

\[
S_n = \frac{n}{2}(a_1 + a_n)
\]

**Exercise #3:** Which of the following is the sum of the first 100 natural numbers? Show the process that leads to your choice.

1. 5,000
2. 5,100
3. 10,000
4. 5,050

**Exercise #4:** Find the sum of each arithmetic series described or shown below.

(a) The sum of the sixteen terms given by:

\[-10 + (-6) + (-2) + \cdots + 46 + 50.\]

(b) The first term is \(-8\), the common difference, \(d\), is 6 and there are 20 terms

(c) The last term is \(a_{12} = -29\) and the common difference, \(d\), is \(-3\).

(d) The sum \(5 + 8 + 11 + \cdots + 77\).

**Exercise #5:** Kirk has set up a college savings account for his son, Maxwell. If Kirk deposits $100 per month in an account, increasing the amount he deposits by $10 per month each month, then how much will be in the account after 10 years?
ARITHMETIC SERIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following represents the sum of $3 + 10 + \cdots + 87 + 94$ if the arithmetic series has 14 terms?

   (1) 1,358  (3) 679
   (2) 658    (4) 1,276

2. The sum of the first 50 natural numbers is

   (1) 1,275  (3) 1,250
   (2) 1,875  (4) 950

3. If the first and last terms of an arithmetic series are 5 and 27, respectively, and the series has a sum 192, then the number of terms in the series is

   (1) 18  (3) 14
   (2) 11  (4) 12

4. Find the sum of each arithmetic series described or shown below.

   (a) The sum of the first 100 even, natural numbers.
   (b) The sum of multiples of five from 10 to 75, inclusive.

   (c) A series whose first two terms are $-12$ and $-8$, respectively, and whose last term is 124.
   (d) A series of 20 terms whose last term is equal to 97 and whose common difference is five.
5. For an arithmetic series that sums to 1,485, it is known that the first term equals 6 and the last term equals 93. Algebraically determine the number of terms summed in this series.

**APPLICATIONS**

6. Arlington High School recently installed a new black-box theatre for local productions. They only had room for 14 rows of seats, where the number of seats in each row constitutes an arithmetic sequence starting with eight seats and increasing by two seats per row thereafter. How many seats are in the new black-box theatre? Show the calculations that lead to your answer.

7. Simeon starts a retirement account where he will place $50 into the account on the first month and increasing his deposit by $5 per month each month after. If he saves this way for the next 20 years, how much will the account contain in principal?

8. The distance an object falls per second while only under the influence of gravity forms an arithmetic sequence with it falling 16 feet in the first second, 48 feet in the second, 80 feet in the third, etcetera. What is the total distance an object will fall in 10 seconds? Show the work that leads to your answer.

9. A large grandfather clock strikes its bell once at 1:00, twice at 2:00, three times at 3:00, etcetera. What is the total number of times the bell will be struck in a day? Use an arithmetic series to help solve the problem and show how you arrived at your answer.
GEOMETRIC SERIES
COMMON CORE ALGEBRA II

Just as we can sum the terms of an arithmetic sequence to generate an arithmetic series, we can also sum the terms of a geometric sequence to generate a geometric series.

**Exercise #1:** Given a geometric series defined by the recursive formula \( a_i = 3 \) and \( a_n = a_{n-1} \cdot 2 \), which of the following is the value of \( S_5 = \sum_{i=1}^{s} a_i \)?

- (1) 106
- (2) 75
- (3) 93
- (4) 35

The sum of a finite number of geometric sequence terms is less obvious than that for an arithmetic series, but can be found nonetheless. The next exercise derives the formula for finding this sum.

**Exercise #2:** Recall that for a geometric sequence, the \( n \)th term is given by \( a_n = a_1 \cdot r^{n-1} \). Thus, the general form of an geometric series is given below.

\[
S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}
\]

(a) Write an expression below for the product of \( r \) and \( S_n \).

\[ r \cdot S_n = \]

(b) Find, in simplest form, the value of \( S_n - r \cdot S_n \) in terms of \( a_1, r, \) and \( n \).

\[ S_n - r \cdot S_n = \]

(c) Write both sides of the equation in (b) in their factored form.

(d) From the equation in part (c), find a formula for \( S_n \) in terms of \( a_1, r, \) and \( n \).

**Exercise #3:** Which of the following represents the sum of a geometric series with 8 terms whose first term is 3 and whose common ratio is 4?

- (1) 32,756
- (2) 28,765
- (3) 42,560
- (4) 65,535
**SUM OF A FINITE GEOMETRIC SERIES**

For a geometric series defined by its first term, $a_1$, and its common ratio, $r$, the sum of $n$ terms is given by:

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a_1-a_r^n}{1-r}$$

**Exercise #4:** Find the value of the geometric series shown below. Show the calculations that lead to your final answer.

$$6 + 12 + 24 + \cdots + 768$$

**Exercise #5:** Maria places $500 at the beginning of each year into an account that earns 5% interest compounded annually. Maria would like to determine how much money is in her account after she has made her $500 deposit at the end of 10 years.

(a) Determine a formula for the amount, $A(t)$, that a given $500 has grown to $t$-years after it was placed into this account.

(b) At the end of 10 years, which will be worth more: the $500 invested in the first year or the fourth year? Explain by showing how much each is worth at the beginning of the 11th year.

(c) Based on (b), write a geometric sum representing the amount of money in Maria’s account after 10 years.

(d) Evaluate the sum in (c) using the formula above.

**Exercise #6:** A person places 1 penny in a piggy bank on the first day of the month, 2 pennies on the second day, 4 pennies on the third, and so on. Will this person be a millionaire at the end of a 31 day month? Show the calculations that lead to your answer.
GEOMETRIC SERIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Find the sums of geometric series with the following properties:
   (a) $a_1 = 6$, $r = 3$ and $n = 8$
   (b) $a_1 = 20$, $r = \frac{1}{2}$, and $n = 6$
   (c) $a_1 = -5$, $r = -2$, and $n = 10$

2. If the geometric series $54 + 36 + \cdots + \frac{128}{27}$ has seven terms in its sum then the value of the sum is
   (1) $\frac{4118}{27}$
   (2) $\frac{1274}{3}$
   (3) $\frac{1370}{9}$
   (4) $\frac{8241}{54}$

3. A geometric series has a first term of 32 and a final term of $-\frac{1}{4}$ and a common ratio of $-\frac{1}{2}$. The value of this series is
   (1) 19.75
   (2) 16.25
   (3) 22.5
   (4) 21.25

4. Which of the following represents the value of $\sum_{i=0}^{8} 256 \left(\frac{3}{2}\right)^i$? Think carefully about how many terms this series has in it.
   (1) 19,171
   (2) 12,610
   (3) 22,341
   (4) 8,956

5. A geometric series whose first term is 3 and whose common ratio is 4 sums to 4095. The number of terms in this sum is
   (1) 8
   (2) 5
   (3) 6
   (4) 4
6. Find the sum of the geometric series shown below. Show the work that leads to your answer.

\[
27 + 9 + 3 + \cdots + \frac{1}{729}
\]

APPLICATIONS

7. In the picture shown at the right, the outer most square has an area of 16 square inches. All other squares are constructed by connecting the midpoints of the sides of the square it is inscribed within. Find the sum of the areas of all of the squares shown. First, consider the how the area of each square relates to the larger square that surrounds (circumscribes) it.

8. A college savings account is constructed so that $1000 is placed the account on January 1st of each year with a guaranteed 3% yearly return in interest, applied at the end of each year to the balance in the account. If this is repeatedly done, how much money is in the account after the $1000 is deposited at the beginning of the 19th year? Show the sum that leads to your answer as well as relevant calculations.

9. A ball is dropped from 16 feet above a hard surface. After each time it hits the surface, it rebounds to a height that is \(\frac{3}{4}\) of its previous maximum height. What is the total vertical distance, to the nearest foot, the ball has traveled when it strikes the ground for the 10th time? Write out the first five terms of this sum to help visualize.
Mortgages, not just on houses, are large amounts of money borrowed from a bank on which interest is calculated (added on) on a regular (typically monthly) basis. Regular payments are also made on the amount of money owed so that over time the principal (original amount borrowed) is paid off as well as any interest on the amount owed. This is a complex process that ultimately involves geometric series. First, some basics.

Exercise #1: Let's say a person takes out a mortgage for $200,000 and wants to make payments of $1600 each month of pay it off. The bank is going to charge this person 4% nominal yearly interest, applied monthly.

(a) What is the amount owed at the end of the first month? Show the calculations that lead to your answer.

(b) How much of the first month's payment went to paying off the principal? How much of it went to paying interest on the loan? Show your calculations.

<table>
<thead>
<tr>
<th>Amount Towards Principal</th>
<th>Amount Towards Interest</th>
</tr>
</thead>
</table>

(c) Determine the amount owed at the end of the second month. Again, show the calculations that lead to your answer.

(d) The amount owed at the end of a month actually forms a sequence that can be defined recursively. If \( a_1 = 200,000 \), then define a recursive rule that gives this sequence.

Exercise #2: If a person took out a $150,000 mortgage at 5% yearly interest, why would it be unwise to have monthly payments of $500?
Now, even if we can define the sequence recursively, as in Exercise #1(d), it would be nice to have a formula that would calculate what we owed after a certain number of months explicitly. To do this, we must see a tricky, extended pattern.

**Exercise #3:** Let's go back to our example of the $200,000 mortgage at 4% yearly interest. Remember, we are paying off this mortgage with $1,600 monthly payments (much of which are initially going to interest). Let's see if we can determine how much we still owe after \( n \)-payments. To make our work easier to follow (and more general), we will let \( r = \frac{.04}{12} = .00312 \), \( P = 200,000 \), and \( m = 1,600 \) to stand for monthly rate (in decimal form, our principal, and our monthly payment).

(a) Explain what the following calculation represents.\[ P(1+r)-m \]

(b) What does the following calculation represent? Write it in expanded form, but leave the binomial \((1+r)\).\[ (P(1+r)-m)(1+r)-m \]

(c) Based on (a) and (b), continue this line of thinking to write expressions for the amount owed at the end of 3 months and 4 months.

(d) Based on (c), write an equation for how much would be owed after \( n \) months.

**Exercise #4:** Now let's see the geometric series. In your answer to (d), you should have the following expression:

\[-m(1+r)^{n-1} - m(1+r)^{n-2} - m(1+r)^{n-3} - \cdots - m(1+r)^2 - m(1+r)-m\]

Given that this is equivalent to: \(-\left(m+m(1+r)+m(1+r)^2+m(1+r)^3+\cdots+m(1+r)^{n-1}\right)\), find the sum of the geometric series inside of the parentheses. Think carefully about the number of terms in this expression.

**Exercise #5:** Find the amount owed on this loan after 10 years or 120 payments (months).
MORTGAGE PAYMENTS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. Consider the mortgage loan of $150,000 at a nominal 6% yearly interest applied monthly at a rate of 0.5% per month. Monthly payments of $1,000 are being made on this loan.

   (a) Determine how much is owed on this loan at the end of the first, second, third month and fourth months. Show the work that leads to your answers. Evaluate all expressions.

   One Month: Two Months:

   Three Months: Four Months:

   (b) The amounts that are owed at the end of each month form a sequence that can be defined recursively. Given that \( a_1 = 150,000 \) represents the first amount owed, give a recursive rule based on what you did in (a) that shows how each successive amount owed depends on the previous one.

   (c) Using a geometric series approach (i.e. the formula we developed in Exercises #3 and #4), determine how much is still owed after 5 years of payments. Show your work.

   (d) Will this loan be paid off after 20 years? What about 30? Provide evidence to support both answers.
When a loan officer speaks to people about a loan for a certain principal, $P$, at a certain monthly rate, $r$, they always have to balance two quantities, the monthly payment, $m$, with the number of payments, $n$, it takes to pay off the loan. These two vary inversely. All of these quantities can be related by the formula:

$$m = \frac{P \cdot r}{1 - (1 + r)^{-n}}$$

This formula is derived by taking the formula we arrived at in Exercises #3 and #4 and setting what we owe equal to zero.

2. Calculate the monthly payment needed to pay off a $200,000 loan at 4% yearly interest over a 20 year period. Recall that $r$ is the monthly rate. Show your work and carefully evaluate the above formula for $m$.

3. Do the same calculation as in the previous exercise but now make the pay off period 30 years instead of 20. How much less is your monthly payment?

It is of interest to also be able to calculate the number of payments (and hence the pay off period) if you have a monthly payment in mind. But this is much more difficult given that you must solve for $n$.

4. Given the formula above:

(a) Show that $n = \frac{-\log\left(1 - \frac{P \cdot r}{m}\right)}{\log(1 + r)}$

(b) Using (a), determine the number of months it would take to pay off a $150,000 loan at a monthly 0.5% rate with $1,000 payments.
UNIT #6

QUADRATIC FUNCTIONS AND THEIR ALGEBRA

Lesson #1 – Quadratic Function Review
Lesson #2 – Factoring
Lesson #3 – Factoring Trinomials
Lesson #4 – Complete Factoring
Lesson #5 – Factoring by Grouping
Lesson #6 – The Zero Product Law
Lesson #7 – Quadratic Inequalities in One Variable
Lesson #8 – Completing the Square and Shifting Parabolas
Lesson #9 – Modeling with Quadratic Functions
Lesson #10 – Equations of Circles
Lesson #11 – The Locus Definition of a Parabola
QUADRATIC FUNCTION REVIEW
COMMON CORE ALGEBRA II

Linear and exponential functions are used throughout mathematics and science due to their simplicity and applicability. Quadratic functions comprise another very important category of functions. You studied these extensively in Common Core Algebra I, but we will review many of their important characteristics in this unit.

**Exercise #1:** Without the use of your calculator, evaluate each of the following quadratic functions for the specified input values. Recall that, according to the formal Order of Operations, exponent evaluation should always come first.

(a) \( f(x) = x^2 \)

(b) \( g(x) = 2x^2 - 5 \)

(c) \( h(x) = -x^2 + 4x \)

\[ f(-3) = \quad g(2) = \quad h(-2) = \]

\[ f(5) = \quad g(-1) = \quad h(3) = \]

Graphs of quadratic functions form what are known as parabolas. The simplest quadratic function, and one that you should be very familiar with, is reviewed in the next exercise.

**Exercise #2:** Consider the simplest of all quadratic functions \( y = x^2 \).

(a) Create a table of values to plot this function over the domain interval \(-3 \leq x \leq 3\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Sketch a graph of this function on the grid to the right.

(c) State the coordinates of the turning point of this parabola.

(d) State the equation of this parabola’s axis of symmetry.

(e) Over what interval is this function increasing?
All quadratic functions that have unlimited domains (domains that consist of the set of all real numbers) have turning points and an axis of symmetry. It is important to be able to sketch a parabola using your graphing calculator to generate a table of values.

**Exercise #3:** Consider the quadratic function \( f(x) = -x^2 + 6x + 5 \).

(a) Using a **TABLE** on your graphing calculator, determine the turning point of this function.

(c) Graph this function on the grid to the right.

(d) Why does this parabola open downward as opposed to \( y = x^2 \) which opened upward?

(e) Between what two consecutive integers does the larger solution to the equation \(-x^2 + 6x + 5 = 0\) lie? Show this point on your graph.

**Exercise #4:** A sketch of the quadratic function \( y = x^2 - 11x - 26 \) is shown below marked with points at its intercepts and its turning point. Using tables or a graph on your calculator, determine the coordinates for each of the points.

The \( x \)-intercepts: \( A \) \( B \)

(Zeroes)

The \( y \)-intercept: \( D \)

The turning point: \( C \)

Over what interval is this function positive?
QUADRATIC FUNCTION REVIEW
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Without the use of your calculator, evaluate each of the following quadratic functions for the specified input values.

   (a) \( g(x) = x^2 - 9 \)
   \[ g(5) = \]
   \[ g(-3) = \]

   (b) \( f(x) = -2x^2 + 8x \)
   \[ f(3) = \]
   \[ f(-1) = \]

   (c) \( h(x) = x^2 - 2x + 6 \)
   \[ h(0) = \]
   \[ h(-2) = \]

2. Which of the following represents the \( y \)-intercept of the graph of the quadratic function \( y = 2x^2 - 7x + 9 \)?  
(Recall, that the \( y \)-intercept of a graph always occurs when \( x = 0 \).)

   (1) 7  
   (2) 2  
   (3) -7  
   (4) 9

3. For a particular quadratic function, the leading coefficient is negative and the function has a turning point whose coordinates are \((-3, 14)\). Which of the following must be the range of this quadratic?

   (1) \( \{y \mid y \geq -3\} \)  
   (2) \( \{y \mid y \leq -3\} \)  
   (3) \( \{y \mid y \leq 14\} \)  
   (4) \( \{y \mid y \geq 14\} \)

4. A parabola has one \( x \)-intercept of \( x = -2 \) and an axis of symmetry of \( x = 4 \). Which of the following represents its other \( x \)-intercept? (Hint, think of how far the given \( x \)-intercept is away from the axis.)

   (1) \( x = 3 \)  
   (2) \( x = 10 \)  
   (3) \( x = 6 \)  
   (4) \( x = 8 \)

5. A quadratic function is shown in the table below Which of the following statements is not true about the function based on this table? Explain your choice.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

(1) The function has an \( x \) intercept of 3.
(2) The function has a \( y \)-intercept of \(-3\).
(3) The function’s leading coefficient is negative.
(4) The function has a turning point of \((1, -4)\)
6. Consider the quadratic function whose equation is \( f(x) = x^2 + 2x - 8 \).

(a) Sketch a graph of \( f \) on the grid provided.

(b) Over what interval is \( f \) decreasing?

(c) Over what interval is \( f(x) < 0 \)?

(d) State the range of \( f \).

APPLICATIONS

7. The number of meters above the ground, \( h \), of a projectile fired at an initial velocity of 86 meters per second and at an initial height of 6.2 meters is given by \( h(t) = -4.9t^2 + 86t + 6.2 \), where \( t \) represents the time, in seconds, since the projectile was fired. If the projectile hits its peak height at \( t = 8.775 \) seconds, which of the following is closest to its greatest height?

(1) 265 meters  (3) 422 meters
(2) 384 meters  (4) 578 meters

8. Physics students were modeling the height of a ball once it was dropped from the roof of a 25 story building. The students found that the height in feet, \( h \), of the ball above the ground as a function of the number of seconds, \( t \), since it was dropped was given by \( h(t) = 300 - 16t^2 \).

From what height was the ball dropped?

To the nearest tenth of a second, determine the time at which the ball hits the ground. Provide evidence from a table to support your answer or solve this algebraically if you recall how to.
FACTORING
COMMON CORE ALGEBRA II

In the study of algebra there are certain skills that are called “gateway skills” because without them a student simply cannot enter into many more complex and interesting problems. Perhaps the most important gateway skill is that of factoring. The definition of factor, in two forms, is given below.

FACTOR – TWO IMPORTANT MEANINGS

1. **Factor** (verb) – To rewrite a quantity as an equivalent product.
2. **Factor** (noun) – Any individual component of a product.

You should be familiar with factoring integers as well as algebraic expressions from earlier courses. We will review some of the basic concepts and techniques of factoring in this lesson.

**Exercise #1**: Factor each of the following integers completely. In other words, write them as the product of only prime numbers (called prime factorization).

(a) 12  
(b) 30  
(c) 16  
(d) 36

**Always** keep in mind that when we factor (verb) a quantity, we are simply rewriting it in a different form that is completely equal to the original quantity. It might look different, but $2 \cdot 3$ is still the number 6.

**Exercise #2**: Rewrite each of the following binomials as a product of an integer with a different binomial.

(a) $5x + 10$  
(b) $2x - 6$  
(c) $6x + 15$  
(d) $6 - 14x$

The above type of factoring is often referred to as “factoring out” the greatest common factor (gcf). This greatest common factor can be comprised of numbers, variables, or both.

**Exercise #3**: Write each of the following binomials as the product of the binomial’s gcf and another binomial.

(a) $3x^2 + 6x$  
(b) $20x - 5x^2$  
(c) $10x^2 + 25x$  
(d) $30x^2 - 20$

**Exercise #4**: Rewritten in factored form $20x^2 - 36x$ is equivalent to

1. $2x(10x - 15)$  
2. $4x(5x - 9)$  
3. $5x(4x + 7)$  
4. $9x(x - 4)$
Trinomials can also sometimes be factored into the product of a gcf and another trinomial.

**Exercise #5:** Rewrite each of the following trinomials as the product of its gcf and another trinomial.

(a) $2x^2 + 8x + 10$  
(b) $10x^2 - 20x + 5$  
(c) $8x^3 - 12x^2 + 20x$  
(d) $6x^3 + 15x^2 - 21x$

Another type of factoring that you should be familiar with stems from our work in the last lesson on conjugates. Recall the conjugate multiplication pattern. This can be “reversed” in order to factor binomials that have the form of the difference of perfect squares.

**CONJUGATE MULTIPLICATION PATTERN**

$$(x-a)(x+a) = x^2 - a^2$$

**Exercise #6:** Write each of the following binomials as the product of a conjugate pair.

(a) $x^2 - 9$  
(b) $4 - x^2$  
(c) $4x^2 - 25$  
(d) $16 - 81x^2$

**Exercise #7:** Write each of the following binomials as the product of a conjugate pair.

(a) $x^2 - \frac{1}{4}$  
(b) $25 - \frac{x^2}{9}$  
(c) $\frac{4}{81}x^2 - \frac{49}{9}$  
(d) $36x^2 - 49y^2$

Factoring an expression until it cannot be factored anymore is known as **complete factoring.** Complete factoring is an important skill to master in order to solve a variety of problems. In general, when completely factoring an expression, the **first** type of factoring always to consider is that of factoring out the gcf.

**Exercise #8:** Using a combination of gcf and difference of perfect squares factoring, write each of the following in its completely factored form.

(a) $5x^2 - 20$  
(b) $28x^2 - 7$  
(c) $40 - 250x^2$  
(d) $3x^3 - 48x$
FACTORS

COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Rewrite each of the following binomials as the product of an integer with a different binomial.
   (a) $10x - 55$
   (b) $24x - 40$
   (c) $6x - 45$
   (d) $18x - 9$

2. Rewrite each of the following binomials as the product of its gcf along with another binomial.
   (a) $2x^2 - 8x$
   (b) $6x + 27$
   (c) $30x^2 - 35x$
   (d) $24x^3 + 20x^2$

3. Rewrite each of the following binomials as the product of a conjugate pair.
   (a) $x^2 - 121$
   (b) $64 - x^2$
   (c) $4x^2 - 1$
   (d) $25x^2 - \frac{1}{9}$

4. Rewrite each of the following trinomials as the product of its gcf and another trinomial.
   (a) $4x^2 + 12x + 28$
   (b) $6x^2 - 4x + 10$
   (c) $14x^3 + 35x^2 - 7x$
   (d) $20x^3 - 5x^2 + 15x$

5. Completely factor each of the following binomials using a combination of gcf factoring and conjugate pairs.
   (a) $6x^2 - 150$
   (b) $36 - 4x^2$
   (c) $28x^2 - 7$
   (d) $27x^3 - 12x$
   (e) $80 - 125x^2$
   (f) $2x^3 - 200x$
   (g) $8x^2 - 512$
   (h) $44x - 99x^3$
6. When completely factored, the expression $48 - 3x^2$ is written as

(1) $3(16 - x)(16 + x)$  
(2) $3(x - 16)(x + 16)$  
(3) $3(x - 4)(x + 4)$  
(4) $3(4 - x)(4 + x)$

7. Which of the following represents the greatest common factor of the terms $4x^2y^6$ and $18xy^5$?

(1) $36xy$  
(2) $4x^2y^3$  
(3) $2xy^5$  
(4) $2x^2y^2$

8. Which of the following is not a factor of $6x^2 - 18x$?

(1) $x - 3$  
(2) $2$  
(3) $12$  
(4) $x$

9. Which of the following prime numbers is not a factor of the integer 330?

(1) 11  
(2) 7  
(3) 3  
(4) 5

APPLICATIONS

10. The area of any rectangular shape is given by the product of its width and length. If the area of a particular rectangular garden is given by $A = 15x^2 - 35x$ and its width is given by $5x$, then find an expression for the garden’s length. Justify your response.

11. The volume of a particular rectangular box is given by the equation $V = 50x - 2x^3$. The height and length of the box are shown on the diagram below. Find the width of the box in terms of $x$. Recall that $V = L\cdot W \cdot H$ for a rectangular box.

12. A projectile is fired from ground level such that its height, $h$, as a function of time, $t$, is given by $h = -16t^2 + 80t$. Written in factored form this equation is equivalent to

(1) $h = -16t(t + 4)$  
(2) $h = -8t(2t - 7)$  
(3) $h = -16t(t - 5)$  
(4) $h = -8t(t - 5)$
Factoring trinomials, expressions of the form \( ax^2 + bx + c \), is an important skill. Trinomials can be factored if they are the product of two binomials. The two main keys to factoring trinomials are: (1) the ability to quickly and accurately multiply binomials (FOIL) and (2) the ability to work with signed numbers. We practice both of these skills with four warm-up multiplication problems in Exercise #1.

**Exercise #1:** Without using your calculator, write each of the following products in simplest \( ax^2 + bx + c \) form.

(a) \((3x + 2)(5x + 7)\)  
(b) \((2x - 3)(2x + 5)\)  
(c) \((5x - 4)(x - 2)\)  
(d) \((4x + 3)(3x - 8)\)

It is important that you know the fundamental rules governing the multiplication and addition of signed numbers. These rules will be key in factoring quickly and correctly. In each case, where a trinomial can be factored, it will be done using the guess-and-check method, where we **intelligently** guess binomial pairs and then check by seeing if the linear terms of the multiplication combine to form the linear term of the trinomial.

**Exercise #2:** Consider the trinomial \( 6x^2 - 35x - 6 \). Below are four guesses of how this trinomial factors.

\[(3x + 2)(2x - 3)\] \[ (x - 3)(x + 2) \] \[ (6x + 1)(x - 6) \] \[ (3x - 2)(2x - 3) \]

(a) Two of these guesses are “unintelligent” – meaning that they should not even be checked. Cross them out and explain below them why they are unreasonable.

(b) Of the two that remain, check both above and determine which is the correct factorization of the trinomial.

The easiest of all trinomial factoring occurs when the leading coefficient is one \( a = 1 \).  

**Exercise #3:** Using a guess-and-check technique, factor each of the following trinomials.

(a) \( x^2 + 2x - 35 \)  
(b) \( x^2 + 11x + 24 \)  
(c) \( x^2 - 13x + 22 \)  
(d) \( x^2 - 5x - 50 \)
A step up from the last exercise occurs when the leading coefficient isn’t one but is still a prime number. This is very often the case and makes at least part of the guessing much easier.

**Exercise #3:** Using a guess-and-check technique, factor each of the following trinomials that have prime leading coefficients. Show each guess and its check.

(a) \(3x^2 + 19x - 40\)  
(b) \(2x^2 - 15x + 18\)

Finally, the hardest trinomials to factor are those whose leading coefficients are not prime. This is due to the fact that there are so many more intelligent guesses. In future lessons we will develop ways to eliminate some of these, but for now, the key will be to just keep **guessing until you get it right**.

**Exercise #4:** Factor each of the following trinomials. Show each guess and its check.

(a) \(15x^2 + 13x + 2\)  
(b) \(10x^2 + 13x - 30\)

(c) \(12x^2 + 8x - 15\)  
(d) \(36x^2 - 35x + 6\)
**Factoring Trinomials**

**Common Core Algebra II Homework**

**Fluency**

1. Multiply each of the following binomial pairs and express your answer in simplest trinomial form.
   
   (a) $(2x+5)(3x-2)$  
   (b) $(3x-8)(5x-1)$  
   (c) $(8x+3)(x+7)$  
   (d) $(7x-5)(5x+2)$

2. Which of the following is the correct factorization of the trinomial $12x^2 - 23x + 10$? Hint – eliminate two of the choices because they are “unintelligent” guesses.
   
   (1) $(6x-1)(3x-10)$  
   (3) $(4x-5)(3x+2)$  
   (2) $(6x-2)(2x-5)$  
   (4) $(4x-5)(3x-2)$

3. Written in factored form $x^2 + 16x - 36$ is equivalent to
   
   (1) $(x-3)(x+12)$  
   (3) $(x-2)(x+18)$  
   (2) $(x-6)(x+6)$  
   (4) $(x-9)(x+4)$

4. Write each of the following trinomials in its factored form. These are the easiest trinomials to factor because the leading coefficient is equal to one.
   
   (a) $x^2 - 7x - 18$  
   (b) $x^2 + 14x + 24$  
   (c) $x^2 - 17x + 30$  
   (d) $x^2 - 5x - 6$

   (e) $x^2 - 5x + 6$  
   (f) $x^2 - 15x + 44$  
   (g) $x^2 + 21x + 20$  
   (h) $x^2 - 6x - 16$
5. Each of the following trinomials has a leading coefficient that is prime. Using a guess-and-check technique, write each trinomial in its factored form. Show each guess and its check.

(a) \(5x^2 - 41x + 8\)

(b) \(3x^2 + 4x - 20\)

(c) \(2x^2 - 29x - 15\)

(d) \(7x^2 + 39x + 20\)

6. Each of the following trinomials has a non-prime leading coefficient. Using a guess-and-check technique, write each trinomial in its factored form. Show each guess and its check.

(a) \(18x^2 - 25x + 8\)

(b) \(20x^2 - 11x - 42\)

**Reasoning**

7. Consider the trinomial \(12x^2 + 7x - 10\).

(a) Does this trinomial have a greatest common factor that could be “factored out”?

(b) Why is \((4x - 2)(3x + 5)\) not an intelligent guess for factoring this trinomial even though \(4 \cdot 3 = 12\) and \(-2 \cdot 5 = -10\)? Consider your answer to part (a).
Each expression that we have factored has been the product of two quantities. But, factoring can produce many more than just two factors. In Exercise #1, we first warm-up by multiplying three factors together.

**Exercise #1:** Write each of these in their simplest form. The last two should take little time to do.

(a) \( (x + 4)(x + 7) \)  
(b) \( 5(2x - 5)(x + 3) \)  
(c) \( 3(x - 5)(x + 5) \)  
(d) \( 4x(3x - 2)(3x + 2) \)

To completely factor an expression means to write it as a product which includes binomials that contain no greatest common factors (gcf’s).

**Exercise #2:** Consider the trinomial \( 2x^2 - 4x - 6 \).

(a) Verify that both of the following products are correct factorizations of this trinomial.

\[
(2x - 6)(x + 1) \quad (2x + 2)(x - 3)
\]

(b) Why are neither of these completely factored?

(c) Write each of these in completely factored form by factoring out the gcf of each unfactored binomial.

(d) What is true of both complete factorizations you found in part (c)?

In practicality, it is always easiest to completely factor by looking for a gcf of the expression first. Once removed, the factoring then either consists of the difference of perfect squares or standard trinomial techniques.

**Exercise #3:** Write each of the following in its completely factored form. These should be relatively easy.

(a) \( 4x^2 + 12x - 40 \)  
(b) \( 6x^2 - 24 \)  
(c) \( 2x^2 + 20x + 50 \)  
(d) \( 75 - 3x^2 \)
Exercise #4: Completely factor each of the following. These will involve final trinomials that are more difficult to guess-and-check.

(a) $10x^2 + 55x - 105$  
(b) $12x^2 + 57x - 15$

The concept of completely factoring an expression by first removing its gcf leads to a helpful factoring tip when working with the trinomial guess-and-check method. This tip will be developed in the next exercise.

Exercise #5: Consider the trinomial $4x^2 + 5x - 6$.

(a) Does this trinomial have a gcf that can be factored out?
(b) The two products listed below are not reasonable guesses for the factorization of this trinomial. Why?

$$\begin{align*}
(2x - 3)(2x + 2) & \quad (4x + 6)(x - 1)
\end{align*}$$

(c) Could a binomial of the form $(2x - a)(2x + b)$, where $a$ and $b$ are divisors of 6, be a correct guess for the factorization of this trinomial? Why or why not?

(d) Factor this trinomial by intelligently guessing-and-checking.

Exercise #6: Use the intelligent factoring tip developed in Exercise #5 to factor each of the following trinomials. Note that neither has a gcf to begin with.

(a) $6x^2 - 13x + 6$  
(b) $12x^2 + 29x - 8$
COMPLETE FACTORING
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Find each of the following products in their simplest $ax^2 + bx + c$ form.
   
   (a) $5(x - 6)(x - 2)$  
   (b) $3(2x - 1)(2x + 1)$  
   (c) $2x(x + 4)(x + 10)$

2. Write each of the following expressions in their completely factored form. These should be moderately easy to factor.
   
   (a) $2x^2 - 14x - 36$ 
   (b) $5x^2 + 70x + 245$ 
   (c) $3x^2 - 192$

   (d) $6x^3 + 36x^2 - 96x$ 
   (e) $28x - 7x^3$ 
   (f) $8x^2 + 12x - 8$

3. Write each of the following in completely factored form. These will involve *slightly more difficult* final trinomial expressions.
   
   (a) $15x^2 - 110x + 120$ 
   (b) $10x^3 - 26x^2 - 12x$
4. Use the factoring tip developed in Exercise #5 to write each of the following trinomials in its factored form. Note that neither has a gcf that can be first factored out.

(a) \(8x^2 + 67x + 24\)  
(b) \(12x^2 - 20x + 3\)

5. More Practice – Write each of the following expressions in its completely factored form.

(a) \(18x^2 - 39x - 15\)  
(b) \(45x - 20x^3\)

(c) \(8x^2 + 30x + 28\)  
(d) \(90x^3 - 90x^2 + 20x\)

(e) \(27x^2 - 3\)  
(f) \(20x^2 + 112x - 48\)
FACToring BY GROUPING
COMMON CORE ALGEBRA II

You now have essentially three types of factoring: (1) greatest common factor, (2) difference of perfect squares, and (3) trinomials. We can combine gcf factoring with the other two to completely factor quadratic expressions. Today we will introduce a new type of factoring known as factoring by grouping. This technique requires you to see structure in expressions.

Exercise #1: Factor a binomial common factor out of each of the following expressions. Write your final expression as the product of two binomials.

(a) \(x(2x+1) + 7(2x+1)\)
(b) \(5x(x-2) - 4(x-2)\)

(c) \((x+5)(x-7) + (x-7)(x+1)\)
(d) \((2x+8)(x+4) - (x-2)(x+4)\)

Exercise #2: Write the expression \((x+3)(x-4) + 5(x+3)\) as the equivalent product of binomials. Test this equivalency with \(x = 2\).

Some very special polynomials can be factored by taking advantage of the structure we have seen in the last two problems. The key is to do mindful manipulations of expressions so that they remain equivalent but are written as an overall product.

Exercise #3: Consider the expression \(2x^3 - 6x^2 + 5x - 15\). Justify each step below with one of the three major properties of real numbers, i.e. the commutative, associative, or distributive.

\[2x^3 - 6x^2 + 5x - 15 = (2x^2 - 6x^2) + (5x - 15)\]
\[= 2x^2(x-3) + 5(x-3)\]
\[= (x-3)(2x^2 + 5)\]
When we **factor by grouping** we first extract common factors from pairs of binomials in four-term polynomials. If we are **lucky** we are left with another **binomial common factor**.

**Exercise #4:** Use the method of factoring by grouping to completely factor the following expressions.

(a) \(3x^3 + 2x^2 - 27x - 18\)  
(b) \(18x^3 + 9x^2 - 2x - 1\)

(c) \(x^5 + 4x^3 + 2x^2 + 8\)  
(d) \(5x^3 + 10x^2 + 20x + 40\)

**Exercise #5:** Consider the expression \(x^2 + ab - ax - bx\).

(a) How can you rewrite the expression so that the first two terms share a common factor (other than 1)?  
(b) Write this expression as an equivalent product of binomials.

Be careful when you use factoring by grouping. Don't force the method when it does not apply. This can lead to errors.

**Exercise #6:** Consider the expression \(2x^3 + 10x^2 + 7x + 21\). Explain the error made in factoring it. How can you tell that the factoring is incorrect?

\[
2x^3 + 10x^2 + 7x + 21 = 2x^2(x + 5) + 7(x + 3) \\
= (2x^2 + 7)(x + 5 + x + 3) \\
= (2x^2 + 7)(2x + 8)
\]
**COMMON CORE ALGEBRA II, UNIT #6 – QUADRATIC FUNCTIONS AND THEIR ALGEBRA – LESSON #5**

**FACTORING BY GROUPING**

**COMMON CORE ALGEBRA I HOMEWORK**

**FLUENCY**

1. Rewrite each of the following as the product of binomials. Be especially careful on the manipulations that involve subtraction.

   (a) \( x(x+5)+7(x+5) \)

   (b) \( 4x(x-2)-3(x-2) \)

   (c) \((x+10)(x-3)+(x+5)(x-3)\)

   (d) \((2x-7)(x+4)+(x+4)(x+2)\)

   (e) \((4x+3)(2x-1)-(x+2)(2x-1)\)

   (f) \((3x+7)(x+5)-(x+5)(2x-4)\)

2. Max tries to simplify the expression \((5x+2)(x+3)-(2x-3)(x+3)\) as follows:

   \[
   = (5x+2)(x+3)-(2x-3)(x+3)
   = (x+3)(5x+2-2x-3)
   = (x+3)(3x-1)
   \]

   Show using \(x=2\) that this simplification is incorrect. Then, give the correct simplification.

3. Factor each of the following quadratic expressions completely using the method of grouping:

   (a) \(10x^2 + 6x + 35x + 21\)

   (b) \(12x^2 + 3x - 20x - 5\)
4. Factor each of the following cubic expressions completely.

(a) $5x^3 + 2x^2 - 20x - 8$

(b) $18x^3 - 27x^2 - 2x + 3$

(c) $x^3 + 2x^2 - 25x - 50$

(d) $8x^3 + 10x^2 + 12x + 15$

5. Factor each of the following expressions. Rearrange the expressions as needed to produce binomial pairs with common factors.

(a) $x^2 - ac - cx + ax$

(b) $xy + ab + ay + bx$

**REASONING**

6. Consider the expression: $x^3 - 5x^2 - 9x + 45$. Enter this expression on your calculator and find its zeroes. Provide evidence. Then, factor it completely. Do you see the relationship between the factors and the zeroes?
The Zero Product Law
COMMON CORE ALGEBRA II

One of the most important equation solving technique stems from a fact about the number zero that is not true of any other number:

**THE ZERO PRODUCT LAW**

If the product of multiple factors is equal to zero then at least one of the factors must be equal to zero.

The law can immediately be put to use in the first exercise. In this exercise, quadratic equations are given already in factored form.

**Exercise #1:** Solve each of the following equations for all value(s) of \( x \).

(a) \((x + 7)(x - 3) = 0\)  
(b) \((2x - 5)(x - 4) = 0\)  
(c) \(4(3x + 2)(4x - 3) = 0\)

**Exercise #2:** In Exercise #1(c), why does the factor of 4 have no effect on the solution set of the equation?

The Zero Product Law can be used to solve any quadratic equation that is factorable (not prime). To utilize this technique the problem solver must first set the equation equal to zero and then factor the non-zero side.

**Exercise #3:** Solve each of the following quadratic equations using the Zero Product Law.

(a) \(x^2 + 3x - 14 = -2x + 10\)  
(b) \(3x^2 + 12x - 7 = x^2 + 3x - 2\)
**Exercise #4:** Consider the system of equations shown below consisting of a parabola and a line.

\[ y = 3x^2 - 8x + 5 \quad \text{and} \quad y = 4x + 5 \]

(a) Find the intersection points of these curves *algebraically*.

(b) Using your calculator, sketch a graph of this system on the axes to the right. **Be sure to label the curves with equations, the intersection points, and the window.**

(c) Verify your answers to part (a) by using the INTERSECT command on your calculator.

The Zero Product Law is extremely important in finding the **zero’s** or **x-intercepts (zeroes)** of a parabola.

**Exercise #5:** The parabola shown at the right has the equation \( y = x^2 - 2x - 3 \).

(a) Write the coordinates of the two \( x \)-intercepts of the graph.

(b) Find the \( x \)-intercepts of this parabola *algebraically*.

**Exercise #6:** *Algebraically* find the set of \( x \)-intercepts (zeroes) for each parabola given below.

(a) \( y = 4x^2 - 1 \) \hspace{1cm} (b) \( y = 3x^2 + 13x - 10 \) \hspace{1cm} (c) \( y = 5x^2 - 10x \)
THE ZERO PRODUCT LAW
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following equations for all value(s) of \( x \).

(a) \((x - 2)(x + 5) = 0\) 
(b) \((7x - 1)(2x + 5) = 0\) 
(c) \((3x - 1)(3x + 1) = 0\)

2. Solve each of the following quadratic equations which have already been set equal to zero.

(a) \(x^2 + 10x + 16 = 0\) 
(b) \(3x^2 + 11x - 4 = 0\) 
(c) \(12x^2 + 8x = 0\)

3. Solve each of the following quadratic equations by first manipulating them so that one side of the equation is set equal to zero.

(a) \(x^2 + 4x - 40 = 10x + 15\) 
(b) \(4x^2 + 3x - 11 = 3x - 2\)

(c) \(6x^2 - 15x + 2 = 2x^2 + 10x - 4\) 
(d) \(-16t^2 + 76t + 5 = 12t + 5\)
APPLICATIONS

4. Consider the system of equations shown below consisting of one linear and one quadratic equation.

\[ y = 4x - 5 \quad \text{and} \quad y = 2x^2 - 5x - 10 \]

(a) Find the intersection points of this system algebraically.

(b) Using your calculator, sketch a graph of this system to the right. Be sure to label the curves with equations, the intersection points, and the window.

(c) Use the INTERSECT command on your calculator to verify the results you found in part (a).

5. Algebraically, find the zeroes (x-intercepts) of each quadratic function given below.

(a) \( y = x^2 - 81 \)  
(b) \( y = 12x^2 - 18x \)  
(c) \( y = 2x^2 - 6x - 8 \)

REASONING

6. A quadratic function of the form \( y = x^2 + bx + c \).

(a) What are the x-intercepts of this parabola?

(b) Based on your answer to part (a), write the equation of this quadratic function first in factored form and then in trinomial form.
QUADRATIC INEQUALITIES IN ONE VARIABLE
COMMON CORE ALGEBRA II

At the heart of solving any inequality is finding all values of the variable (or variables) that make the inequality true. This basic notion of inequalities is critical to understand before proceeding.

**Exercise #1:** Determine if each of the following is a solution to the inequality given. Show work to justify your response.

(a) \( x^2 - 3x - 10 > 0 \) for \( x = 4 \)
(b) \( 2x^2 + 13x - 7 \geq 0 \) for \( x = 2 \)
(c) \( x^2 - x - 12 < 0 \) for \( x = -3 \)

Most of the time, there are an infinite number of solutions to an inequality. The solution set of inequalities like these cannot be written in roster form (where one lists the solutions). In **Exercise #2**, we will explore how to determine this solution set by using tables on your calculator.

**Exercise #3:** Consider the quadratic inequality \( x^2 + 2x - 3 < 0 \).

(a) Solve the corresponding equation \( x^2 + 2x - 3 = 0 \) algebraically for all values of \( x \).
(b) Using your calculator and the equation \( y = x^2 + 2x - 3 \) fill in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Explain why the zeroes you found in part (a) are **not** part of the solution set of the inequality.
(d) Write the solution set of the inequality and represent it on a number line.

The key to algebraically solving a quadratic inequality is to first find the zeroes and then test points between the zeroes and outside the zeroes.

**Exercise #3:** Which of the following is the solution set of the inequality \( x^2 - 4 > 0 \)?

(1) \( \{ x \mid x > 2 \} \)  
(2) \( \{ x \mid -2 < x < 2 \} \)  
(3) \( \{ x \mid x > 2 \text{ or } x < -2 \} \)  
(4) \( \{ x \mid x > -2 \} \)
**Exercise #4:** Solve each of the following quadratic inequalities. Write your final answers in set-builder notation and represent the solution set on a number line.

(a) \( x^2 - 5x - 36 \leq 0 \)  
(b) \( 5x^2 + 28x - 12 > 0 \)

(c) \( 2x^2 - 4x - 8 \geq 10x - 8 \)  
(d) \( x^2 + 14x - 6 < 14x + 19 \)

---

**Exercise #5:** The number line graph \(-5 \rightarrow 0 \rightarrow 5\) is the solution to which of the following inequalities?

1. \( x^2 - 2x - 8 > 0 \)
2. \( x^2 + 2x - 8 < 0 \)
3. \( x^2 - 2x - 8 \geq 0 \)
4. \( x^2 + 2x - 8 \leq 0 \)

---

**Exercise #6:** Which of the following represents the solution set of the inequality \(-2x^2 + 7x - 3 > 0\)?

1. \( \{x \mid \frac{1}{2} < x < 3\} \)
2. \( \{x \mid -\frac{1}{2} < x < 3\} \)
3. \( \{x \mid x < -3 \text{ or } x > \frac{1}{2}\} \)
4. \( \{x \mid x < \frac{1}{2} \text{ or } x > 3\} \)
QUADRATIC INEQUALITIES IN ONE VARIABLE
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following values of \( x \) is in the solution set of the inequality \( x^2 + x - 2 > 0 \)? Hint – to make this problem easier, generate a table on your calculator using \( y = x^2 + x - 2 \).

   (1) 1   (3) 0
   (2) -2  (4) -4

2. Which of the following values of \( x \) is not in the solution set of the inequality \( 5x^2 + 35x \leq 0 \)?

   (1) -1   (3) 0
   (2) 2    (3) -7

3. The solution set of the inequality \( x^2 > 25 \) is which of the following?

   (1) \((5, \infty)\)   (3) \((-\infty, -5) \cup (5, \infty)\)
   (2) \([-5, 5]\)    (4) \((-\infty, 5]\)

4. The solution to the inequality \( x^2 - 9 < 0 \) can be expressed graphically as

   (1) \[
   \begin{array}{c}
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \text{ -5 } & \text{ 0 } & \text{ 5 } \\
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \end{array}
   \]
   (2) \[
   \begin{array}{c}
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \text{ -5 } & \text{ 0 } & \text{ 5 } \\
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \end{array}
   \]
   (3) \[
   \begin{array}{c}
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \text{ -5 } & \text{ 0 } & \text{ 5 } \\
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \end{array}
   \]
   (4) \[
   \begin{array}{c}
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \text{ -5 } & \text{ 0 } & \text{ 5 } \\
   \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
   \end{array}
   \]

5. Which of the following is the solution set of \((x + 5)(x - 3) < 0\)?

   (1) \( \{x \mid -5 < x < 3\} \)  (3) \( \{x \mid x < -5 \text{ or } x > 3\} \)
   (2) \( \{x \mid -5 \leq x \leq 3\} \)  (4) \( \{x \mid -3 < x < 5\} \)

6. Which inequality below represents all solutions to \( x^2 \geq 5x + 24 \)?

   (1) \( \{x \mid -6 \leq x \leq 4\} \)  (3) \( \{x \mid x \leq -8 \text{ or } x \geq 3\} \)
   (2) \( \{x \mid -2 \leq x \leq 12\} \)  (4) \( \{x \mid x \leq -3 \text{ or } x \geq 8\} \)
7. Find the solution set to each of the quadratic inequalities shown below. Represent your solution set using any acceptable notation and graphically on a number line.

(a) \(2x^2 + 9x - 35 < 0\)  
(b) \(x^2 \geq 5x + 6\)

(c) \(8x^2 + 50x - 5 < 10x - 5\)  
(d) \(4x^2 + 23x - 6 \geq 0\)

(e) \(x^2 \leq 10x + 24\)  
(f) \(7x^2 + 4x + 3 > 3x^2 + 4x + 4\)
Completing the Square and Shifting Parabolas
Common Core Algebra II

Parabolas, and graphs more generally, can be moved horizontally and vertically by simple manipulations of their equations. This is known as shifting or translating a graph. You worked with this extensively in Common Core Algebra I. The first exercise will review how to use a method known as completing the square to identify shifts and the turning point of a parabola.

Exercise #1: The function \( y = x^2 \) is shown already graphed on the grid below. Consider the quadratic whose equation is \( y = x^2 - 8x + 18 \).

(a) Using the method of completing the square, write this equation in the form \( y = (x-h)^2 + k \).

(b) Describe how the graph of \( y = x^2 \) would be shifted to produce the graph of \( y = x^2 - 8x + 18 \).

(c) Sketch the graph of \( y = x^2 - 8x + 18 \) by using its vertex form in (a). What are the coordinates of its turning point (vertex)?

Exercise #2: Using your calculator and the window shown below, sketch the graphs of the simple quadratics \( y = x^2, y = 3x^2, \) and \( y = \frac{1}{2} x^2 \).

Every quadratic of the form \( y = ax^2 \) has a turning point at:
The algorithm of completing the square works best when $a = 1$ and $b$ is even in the form $y = ax^2 + bx + c$. But, it does work in every case, even the messy ones.

**Exercise #3:** Place each of the following quadratic functions in vertex form and identify the turning point.

(a) $y = 3x^2 + 12x - 2$  
(b) $y = 2x^2 + 6x + 1$

**Exercise #4:** The method of completing the square can be performed on the standard quadratic equation $y = ax^2 + bx + c$ and after much manipulation can be placed in the form:

$$y = a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c$$

(a) Based on this formula, what is the $x$-coordinate of the turning point of any parabola? Be careful.  
(b) Use this formula to find the turning point of the parabola $y = x^2 + 10x - 2$.

(c) Verify your answer from part (a) by placing the quadratic $y = x^2 + 10x - 2$ into vertex form.  
(d) Verify both answers by examining a table on your calculator using the original equation.

**Exercise #5:** Use the formula $x = -\frac{b}{2a}$ to find the turning points for each of the following quadratic functions.

(a) $f(x) = 2x^2 - 12x + 7$  
(b) $g(x) = -\frac{1}{4}x^2 + 5x - 20$
COMPLETING THE SQUARE AND SHIFTING PARABOLAS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following equations would result from shifting $y = x^2$ five units right and four units up?

   (1) $y = (x - 5)^2 + 4$ 
   (3) $y = (x - 4)^2 - 5$

   (2) $y = (x + 5)^2 + 4$ 
   (4) $y = (x + 4)^2 - 5$

2. Which of the following represents the turning point of the parabola whose equation is $y = (x + 3)^2 - 7$?

   (1) $(3, -7)$ 
   (3) $(-7, -3)$

   (2) $(-3, 7)$ 
   (4) $(-3, -7)$

3. Which of the following quadratic functions would have a turning point at $(6, -2)$?

   (1) $y = (x + 6)^2 - 2$ 
   (3) $y = 5(x - 6)^2 - 2$

   (2) $y = 3(x + 2)^2 - 2$ 
   (4) $y = 2(x - 1)^2 + 6$

4. Which of the following is turning point of $y = x^2 + 12x - 4$?

   (1) $(12, -4)$ 
   (3) $(6, 104)$

   (2) $(-6, -40)$ 
   (4) $(-4, 12)$

5. In vertex form, the parabola $y = x^2 - 10x + 8$ would be written as

   (1) $y = (x - 5)^2 - 33$ 
   (3) $y = (x - 10)^2 - 92$

   (2) $y = (x - 5)^2 - 17$ 
   (4) $y = (x - 10)^2 - 108$

6. The turning point of the parabola $y = x^2 + 5x - 2$ is

   (1) $(2.5, 12.75)$ 
   (3) $(-2.5, -8.25)$

   (2) $(-5, -10.5)$ 
   (4) $(-2.5, -17.5)$
7. Write each of the following quadratic functions in its vertex form by completing the square. Then, identify its turning point.

(a) \( y = x^2 + 12x + 50 \)  
(b) \( y = -3x^2 + 30x + 7 \)

8. Use the formula \( x = -\frac{b}{2a} \) to find the turning points of each of the following quadratic functions. Then, place the function in vertex form to verify the turning points.

(a) \( y = 5x^2 - 30x + 55 \)  
(b) \( y = -2x^2 - 24x - 67 \)

9. Consider the quadratic function whose equation is \( y = x^2 + 6x - 40 \).

(a) Determine the \( y \)-intercept of this function algebraically.  
(b) Write the function in its vertex form. State the coordinates of its turning point.

(c) Algebraically find the zeroes of the function using the zero product law.  
(d) Sketch a graph of the parabola, showing all relevant features found in parts (a) through (c).
We have already seen quadratic functions used to model a wide variety of real-world applications. In this lesson we will focus on using quadratic models to predict and optimize. Almost always our solution will depend on either the intercepts or the turning point of the quadratic model.

**Exercise #1:** An object is fired upwards with an initial velocity of 112 feet per second. Its height, in feet above the ground, as a function of time, in seconds since it was fired, is given by the equation \( h(t) = -16t^2 + 112t \).

(a) At what height was the object fired? 

(b) Sketch a general curve of this equation below.

(c) Algebraically, find the time that the rocket reaches its greatest height and the maximum height. Label these on the graph that you drew in part (b).

(d) Algebraically, determine the time when the rocket reaches the ground. Label this on your graph in (b).

**Exercise #2:** A skateboard half-pipe ramp has a shape in the form of a parabola whose equation is \( y = 0.06x^2 - 1.2x + 7 \) where \( x \) represents the horizontal distance across the 20-foot wide half-pipe and \( y \) represents the ramp’s height above the ground in feet. With the help of your calculator, sketch a graph of the half-pipe below. Label its height at its endpoints and its minimum point.
Exercise #3: The Crazy Carmel Corn company has determined that the percentage of kernels that pop rises and then falls as the temperature of the oil the kernels are cooked in increases. It modeled this trend using the equation

\[ P = -\frac{1}{250}T^2 + 2.8T - 394 \]

where \( P \) represents the percent of the kernels that pop and \( T \) represents the temperature of the oil in degrees Fahrenheit.

(a) Algebraically determine the temperature at which the highest percentage of kernels pop. Also, determine the percent of kernels that pop at this temperature.

(b) Using your calculator, sketch a curve below for \( P \geq 0 \). Label your window.

(c) Using the ZERO command on your calculator, determine, to the nearest degree, the two temperatures at which \( P = 0 \). Label them on your graph drawn in part (b).

(d) If a typical batch of popcorn consists of 800 kernels, how many does the Crazy Carmel Corn company expect to pop at the optimal temperature?

(e) For a batch of popcorn to be successful, the company wants at least 85% of its kernels to pop. Write an inequality whose solution represents all temperatures that would ensure a successful batch. Solve this inequality graphically, to the nearest degree, and show your graph to below, labeling all relevant points.
MODELING WITH QUADRATIC FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. The height of a missile \( t \) seconds after it has been fired is given by \( h = -4.9t^2 + 44.1t \). Which of the following represents the number of seconds it will take for the rocket to reach its greatest height?

   (1) 108  (3) 99
   (2) 4.5  (4) 7.5

2. The daily cost per car manufactured at a certain automotive plant decreases as the number of cars increase and then increases again due to overtime production costs. The cost \( C \), per car, is given by \( C(n) = 0.3n^2 - 90n + 12,450 \) where \( n \) represents the number of cars produced. Which of the following is the lowest per car cost?

   (1) $5,700  (3) $12,450
   (2) $150  (4) $2,150

3. A decathlete at the Olympics throws a javelin such that its height, \( h \), above the ground can be modeled as a quadratic function of the horizontal distance, \( d \), that it has traveled. Which of the following is a realistic quadratic function for this scenario?

   (1) \( h = \frac{1}{100}d^2 + 75d + 3 \)
   (3) \( h = -\frac{1}{100}d^2 + 75d + 3 \)
   (2) \( h = \frac{1}{100}d^2 + 75d - 3 \)
   (4) \( h = -\frac{1}{100}d^2 + 75d - 3 \)

4. A ball thrown vertically in the air reaches its peak height after 3.5 seconds. If its height, as a function of time, is given by \( h = -16t^2 + bt + 4 \), then which of the following is the value of \( b \)?

   (1) 56  (3) -112
   (2) -56  (4) 112

5. A tour company has a ticket price that goes down $2 for every additional person who signs up for a group trip. They charge, per person, \( 52 - 2n \) where \( n \) is the number of people that go on the trip. Their total revenue, \( R \), as a function of the number of people who go on the trip is \( R = 52n - 2n^2 \). How many people maximize the revenue for the tour company?

   (1) 13  (3) 26
   (2) 39  (4) 22
6. Bacteria tend to grow very fast in a Petri dish at first because of unlimited food and then begin to die out due to competition. In a certain culture, the number of bacteria is given by \( N(t) = -2t^2 + 92t + 625 \), where \( t \) represents the hours since 625 bacteria were introduced to the Petri dish. Determine the maximum number of bacteria that occur in the dish.

7. A tennis ball is thrown upwards from the top of a 30-foot high building. Its height, in feet above the ground, \( t \)-seconds after it is thrown is given by \( h = -16t^2 + 80t + 30 \).

(a) Algebraically determine the time when the tennis ball reaches its greatest height. What is that height?

(b) Using your calculator, sketch a general graph showing the ball’s height for all times where \( t \geq 0 \) and \( h \geq 0 \). Label the information you found in part (a).

(c) Using the ZERO command on your calculator, determine the amount of time the ball stays in the air. Round your answer to the nearest tenth of a second and label this on your graph drawn in part (a).

(d) The ball can be seen from the ground whenever it is at a height of at least 100 feet. Graphically determine the interval of time that the ball can be seen. Show the work on your graph from in part (b).

8. The area of a rectangle whose perimeter is a fixed 80 feet is given by \( A = 40w - w^2 \), where \( w \) is the width of the rectangle. Determine the width of the rectangle that gives the maximum area. What type of special rectangle is necessary to produce this maximum area? Justify.
EQUATIONS OF CIRCLES
COMMON CORE ALGEBRA II

Various quadratic relationships can be placed into equations by knowing the locus definition of the relationship. We will explore this for parabolas in a future lesson. In this one, we will develop the equation of a circle by using the distance formula that you learned from Common Core Geometry.

THE DISTANCE FORMULA

The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by: 

\[ D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Exercise #1: A circle is the collection of all points that are a set distance (the radius) away from a point (its center). The circle shown below has a radius of 5 and a center at the point \((4, 2)\). An arbitrary point on the circle, \((x, y)\), is shown marked.

(a) Using the distance formula show that the point \((7, -2)\) must lie on this circle (verify graphically).

(b) Letting \((x_2, y_2) = (x, y)\) and \((x_1, y_1) = (4, 2)\), write the distance formula for all points on this circle.

(c) Square both sides of the equation from (b) to create the standard form of a circle.

(d) Show algebraically that the point \((-1, 2)\) must also lie on the circle.

THE EQUATION OF A CIRCLE

A circle whose center is at \((h, k)\) and whose radius is \(r\) is given by: 

\[ (x - h)^2 + (y - k)^2 = r^2 \]

Exercise #2: Which of the following equations would have a center of \((-3, 6)\) and a radius of 3?

1. \((x - 3)^2 + (y + 6)^2 = 9\)  
2. \((x + 3)^2 + (y - 6)^2 = 9\)

3. \((x - 3)^2 + (y - 6)^2 = 3\)  
4. \((x + 3)^2 + (y + 6)^2 = 3\)
Exercise #3: For each of the following equations of circles, determine both the circle’s center and its radius. If its radius is not an integer, express it in decimal form rounded to the nearest tenth.

(a) \((x - 2)^2 + (y - 7)^2 = 100\)  
(b) \((x - 5)^2 + (y + 8)^2 = 4\)  
(c) \(x^2 + y^2 = 121\)

(d) \((x + 1)^2 + (y + 2)^2 = 1\)  
(e) \(x^2 + (y - 3)^2 = 49\)  
(f) \((x + 6)^2 + (y - 5)^2 = 18\)

(g) \(x^2 + y^2 = 64\)  
(h) \((x - 4)^2 + (y - 2)^2 = 20\)  
(i) \(x^2 + y^2 = 57\)

Exercise #4: Write equations for circles \(A\) and \(B\) shown below. Show how you arrive at your answers.

Exercise #5: By completing the square on both quadratic expressions in \(x\) and \(y\) determine the center and radius of a circle whose equation is

\[x^2 + 10x + y^2 - 2y = 10\]
**EQUATIONS OF CIRCLES**
**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

1. Each of the following is an equation of a circle. State the circle’s center and radius. In the cases where the radius is not an integer, give its value rounded to the nearest tenth.

   (a) \( x^2 + y^2 = 144 \)  
   (b) \( (x - 3)^2 + (x + 7)^2 = 36 \)  
   (c) \( (x + 5)^2 + (y + 1)^2 = 64 \)  

   (d) \( (x - 2)^2 + (y - 9)^2 = 100 \)  
   (e) \( x^2 + y^2 = 1 \)  
   (f) \( x^2 + (y + 5)^2 = 25 \)  

   (g) \( x^2 + y^2 = 50 \)  
   (h) \( (x - 3)^2 + y^2 = 200 \)  
   (i) \( (x - 6)^2 + (y + 6)^2 = 20 \)

2. Which of the following is true about a circles whose equation is \( (x + 5)^2 + (y - 3)^2 = 36 \)?

   (1) It has a center of \( (5, -3) \) and an area of \( 12\pi \).
   (2) It has a center of \( (-5, 3) \) and a diameter of 6.
   (3) It has a center of \( (-5, 3) \) and an area of \( 36\pi \).
   (4) It has a center of \( (5, -3) \) and a circumference of \( 12\pi \).

3. Which of the following represents the equation of the circle shown graphed below?

   (1) \( (x - 2)^2 + (y + 3)^2 = 16 \)
   (2) \( (x + 2)^2 + (y - 3)^2 = 4 \)
   (3) \( (x - 2)^2 + (y + 3)^2 = 4 \)
   (4) \( (x + 2)^2 + (y - 3)^2 = 16 \)

4. By completing the square on each of the quadratic expressions, determine the center and radius of a circle whose equation is shown below.

   \( x^2 - 6x + y^2 + 10y = 66 \)
5. Circles are described below by the coordinates of their centers, $C$, and one point on their circumference, $A$. Determine an equation for each circle in center-radius form.

(a) $C(5, 2)$ and $A(11, 10)$  
(b) $C(-2, -5)$ and $A(3, -17)$  
(c) $C(5, -1)$ and $A(-2, -5)$

6. Solve the following system of equations graphically.

\[ x^2 + y^2 = 25 \]
\[ y = 5 - x^2 \]

7. Find the intersection of the circle $x^2 + y^2 = 29$ and $y = x - 3$ algebraically.

**APPLICATIONS**

7. Jonas is designing a circular garden whose equation is $x^2 + y^2 = 49$. He wishes to place a walkway within the garden at all points within the circle that satisfy the inequality $-2 \leq y \leq 2$. Graph the circle on the grid to the right and shade in all points that represent the walkway.
The circle had a relatively easy locus definition, i.e. the collection of all points **equidistant** from a given point. Parabolas have a slightly more complex definition that we will explore in this lesson.

**THE LOCUS DEFINITION OF A PARABOLA**

A parabola is the collection of all points **equidistant** from a fixed point (known as its **focus**) and a fixed line (known as its **directrix**).

**Exercise #1:** The parabola \( y = \frac{1}{4} x^2 + 1 \) is shown graphed below with selected points shown. For this parabola, its focus is the point \((0, 2)\) and its directrix is the \(x\)-axis.

(a) How far is the turning point \((0, 1)\) from both the focus and directrix? How far is the point \((2, 2)\) from both?

(b) Use the distance formula to verify that the point \((4, 5)\) is the same distance away from the focus and directrix. Draw line segments from the focus and directrix to this point to visualize the distance. Repeat for the point \((6, 10)\)

(c) Use the distance formula to show that the equation of this parabola is \( y = \frac{1}{4} x^2 + 1 \) based on the locus definition of a parabola.
The algebra for finding the equation of a parabola based on its focus and directrix can be challenging, but you know all of it from previous work you have done. Just be careful with each step.

Exercise #2: Consider a parabola whose focus is the point \((0, 7)\) and whose directrix is the line \(y = 3\).

(a) Sketch a diagram of the parabola below and identify its turning point.

(b) Determine the equation of the parabola using the locus definition.

Exercise #3: Determine the equation of the parabola whose focus is the point \((4, 1)\) and whose directrix is the horizontal line \(y = -3\). First, draw a diagram that shows the parabola, then carefully use the distance formula to derive its equation.
THE LOCUS DEFINITION OF A PARABOLA
COMMON CORE ALGEBRA I HOMEWORK

FLUENCY

1. Fill in the following locus definition of a parabola with one of the words shown listed below. Words may be used more than once.

point, line, equidistant, directrix, collection, focus

A parabola is the ________________ of all points ________________ from a fixed ________________ and a fixed ________________.

The fixed ________________ is known as the parabola's ________________.

The fixed ________________ is known as the parabola's ________________.

2. The parabola whose equation is \( y = \frac{1}{8}x^2 + 2 \) is shown graphed on the grid below. Its directrix is the \( x \)-axis.

(a) Explain why the focus must be the point \((0, 4)\).
Label this point on the diagram.

(b) How far is the point \((4, 4)\) from both the focus and the directrix?

(c) Show that the point \((8, 10)\) is equidistant from the focus and directrix.

(d) Using the locus definition of a parabola, show that the equation is \( y = \frac{1}{8}x^2 + 2 \).
2. Consider parabola that is the collection of all points equidistant from the point \((0, 8)\) and the line \(y = 2\).

- **(a)** Give each of the following:
  - **Directrix:**
  - **Focus:**

- **(c)** Find the equation of this parabola using the locus definition.

![Diagram of a parabola](image)

3. Parabolas can be constructed using the classic geometric tools of a compass and a straightedge. The circles below represent all the points equidistant from the focus \((0, 4)\). Given this focus point and a directrix of the \(x\)-axis, do the following.

- **(a)** Draw in the horizontal lines \(y = 2, y = 3, y = 4, y = 5, y = 6, y = 7,\) and \(y = 8\). These lines represent points that are a fixed distance away from the \(x\)-axis (the directrix).

- **(b)** Plot points at the intersections of the lines your drew in (a) with the circles that are the same distance from the focus. Connect with a smooth parabolic curve.

![Diagram of a parabola](image)
UNIT #7

TRANSFORMATIONS OF FUNCTIONS

Lesson #1 – Shifting Functions
Lesson #2 – Reflecting Parabolas
Lesson #3 – Vertically Stretching Functions
Lesson #4 – Horizontal Stretching Functions
Lesson #5 – Even and Odd Functions
SHIFTING FUNCTIONS
COMMON CORE ALGEBRA II

The basic geometric transformations of translating (shifting), reflecting, rotating, and dilating can all be done to the graphs of functions by algebraically manipulating them. Rotating functions is out of the scope of our work in this course, but we will investigate all of the others, all of which you originally saw in Common Core Algebra I. In this lesson, we will concentrate on shifting functions.

Exercise #1: Consider the functions \( y = |x|, \ y = |x - 3| + 2, \) and \( y = |x + 1| - 4. \)

(a) Without the use of your calculator, graph \( y = |x| \) on the axes provided. Label its equation.

(b) Using your calculator to generate a table of values, graph the other two absolute value functions above and label each with its equation.

(c) How would the graph of \( y = |x| \) be shifted in order to produce the graph of \( y = |x - 6| - 8? \)

Although we just used the absolute value function, this exercise illustrates the fundamental manner in which vertical shifts and horizontal shifts occur in functions.

VERTICAL AND HORIZONTAL SHIFTING

1. **Vertical Shifting**: The function \( f(x) + k \) shifts the function up by \( |k| \) units for \( k > 0 \) and down \( |k| \) units for \( k < 0 \).

2. **Horizontal Shifting**: The function \( f(x + k) \) shifts the function left \( |k| \) units for \( k > 0 \) and right \( |k| \) units for \( k < 0 \).

Exercise #2: The function \( f(x) \) is shown on the grid below. A second function, \( g(x) = f(x - 3) + 1 \).

(a) What is the value of \( g(0) \)? Show how you arrived at your answer.

(b) Identify how the graph of \( f \) has been transformed to produce the graph of \( g \) and sketch it on the grid.
We can use these shifting patterns in a variety of ways because they apply to all types of functions.

**Exercise #3:** A function, \( f(x) \), has a domain of \(-3 \leq x \leq 10\) and a range of \( y \leq 22\). What are the domain and range of the function \( f(x+7)+10 \)? Explain how you arrived at your answers.

Recognizing shifts of other, simpler functions can help us identify prominent characteristics and compare them. The location of turning points is especially helpful.

**Exercise #4:** Given the quadratic function \( f(x) = (x-4)^2 - 5 \) answer the following questions.

(a) How has the simple quadratic \( y = x^2 \) been shifted to produce the graph of \( f(x) \)?

(b) Given that \( y = x^2 \) has a turning point at the origin, \((0, 0)\), where must the turning point of \( f \) lie?

(c) Sketch \( f \) below and give the domain interval over which \( f \) is increasing.

(d) Which has a lower minimum value, the function \( f \) or the function \( g(x) = |x-6|-10 \)? Explain your choice.

One of the hardest things for students to grasp is the horizontal shift, which appears to work opposite of what we would expect. Let's take a look at a shift that is purely horizontal.

**Exercise #5:** The graph of \( f(x) \) is shown below. The function \( g(x) \) is defined by \( g(x) = f(x-2) \).

(a) Show that \( x = -1 \) and \( x = 2 \) are zeroes of the function \( g \).

(b) Evaluate each of the following using the definition of \( g \) and then create a plot of \( g \) on the same set axes.

\[
g(-3) = \quad g(0) = \quad g(4) =
\]
**SHifting Functions**

**Common Core Algebra II Homework**

**Fluency**

1. Given the function \( f(x) \) shown graphed on the grid, create a graph for each of the following functions and label on the grid.

   (a) \( g(x) = f(x) + 2 \)

   (b) \( h(x) = f(x - 3) \)

   (c) \( k(x) = f(x + 1) - 4 \)

2. If the quadratic function \( f(x) \) has a turning point at \((-3, 7)\) then where does the quadratic function \( g \) defined by \( g(x) = f(x + 4) + 5 \) have a turning point?

   (1) \((-7, 12)\)  
   (2) \((1, 12)\)  
   (3) \((-4, 5)\)  
   (4) \((4, 5)\)  

3. Over which of the following intervals would the function \( h(x) = |x - 2| + 6 \) be decreasing only? Sketch a graph of the function if needed.

   (1) \( x > 2 \)  
   (2) \( x < 2 \)  
   (3) \( x < 6 \)  
   (4) \( x > 6 \)  

4. If the domain of \( f(x) \) is \(-3 \leq x \leq 9\) and the range of \( f(x) \) is \( 2 \leq y \leq 15 \), then which of the following statements is correct about the domain and range of \( g(x) = f(x - 2) - 8 \)?

   (1) Its domain is \(-1 \leq x \leq 11\) and its range is \( 10 \leq y \leq 23\).

   (2) Its domain is \(-5 \leq x \leq 7\) and its range is \(-6 \leq y \leq 7\).

   (3) Its domain is \(-1 \leq x \leq 11\) and its range is \(-6 \leq y \leq 7\).

   (4) Its domain is \(-5 \leq x \leq 7\) and its range is \(-6 \leq y \leq 23\).
5. The graph of the function \( f(x) \) is shown on the grid below. The function \( g \) is defined by the formula \( g(x) = f(x+3) - 1 \).

(a) Graph and label \( g \) on the axes along with \( f \).

(b) What is the smallest solution to the equation \( f(x) = g(x) \)?

(c) If \( h(x) = g(x) - 3 \), explain why the equation \( h(x) = f(x) \)
has no solutions.

\[ \text{APPLICATIONS} \]

6. A projectile has a height given by the function \( h(t) = -4.9(t-4)^2 + 153 \), where times, \( t \), is in seconds and the height, \( h \), is in meters. What is the maximum height of the function and at what time does it reach that height?

\[ \text{REASONING} \]

7. Given the linear equations \( f(x) = 2x \) and \( g(x) = 2x - 2 \) answer the following.

(a) Show that the function \( f \) passes through the origin.

(b) How has the function \( f \) been shifted to produce the function \( g \)?

(c) Write the function \( g \) in factored form.

(d) Based on (c), how has the function \( f \) been shifted to produce the function \( g \)?

(e) How would \( f(x) \) need to be shifted to produce \( h(x) = 2(x+5) - 7 \)? Given that \( f \) must contain the point \((0, 0)\), what point must \( h(x) \) contain based on the shifting?
Reflecting functions across the x and y axes are an important mathematical processes that will be explored in this lesson specifically for parabolas, although the general ideas apply to all types of functions. The first exercise gets at both of these important transformations.

**Exercise #1:** The parabola \( f(x) = x^2 - 6x + 5 \) is shown on the grid below.

(a) Consider the function \( g(x) = -f(x) \). Determine a formula for \( g(x) \) and graph it on the grid below.

(b) How was the graph of \( f \) transformed to produce the graph of \( g \)?

(c) Now consider the function \( h(x) = f(-x) \). Determine a formula for \( h(x) \) and graph it on the grid above.

(d) How was the graph of \( f \) transformed to produce the graph of \( h \)?

---

**Reflecting Functions in the x and y Axes**

The function \( -f(x) \) is a reflection of \( f(x) \) in the x-axis.

The function \( f(-x) \) is a reflection of \( f(x) \) in the y-axis.

**Exercise #2:** Determine an equation for the linear function \( g(x) = 5x - 7 \) both after a reflection in the x-axis and y-axis. Label your equations.
**Exercise #3:** If a parabola has the equation \( f(x) = 2x^2 - 3x + 8 \), which of the following represents its equation after a reflection in the x-axis?

(1) \( y = 2x^2 + 3x + 8 \)  (3) \( y = -2x^2 + 3x + 8 \)
(2) \( y = -2x^2 + 3x - 8 \)  (4) \( y = 2x^2 - 3x - 8 \)

**Exercise #4:** After a reflection in the y-axis, the quadratic function \( g(x) = 4x^2 - 7x + 2 \) would have the equation

(1) \( y = -4x^2 + 7x + 2 \)  (3) \( y = 4x^2 + 7x + 2 \)
(2) \( y = -4x^2 + 7x - 2 \)  (4) \( y = 4x^2 + 7x - 2 \)

**Exercise #5:** Consider the function \( g(x) = -x^2 + 4 \). What two transformations have occurred to the graph of \( y = x^2 \) to produce the graph of \( g \)? Specify the transformations and the order in which they occurred. Note that there exists more than one correct answer. Graph on your calculator to verify.

**Exercise #6:** The graph of a function \( f(x) \) is shown below on two grids. Sketch (a) the graph of \(-f(x)\) and (b) the graph of \( f(-x) \).

(a) Graph and label \(-f(x)\).

(b) Graph and label \( f(-x) \).
**Reflecting Parabolas**

**Common Core Algebra II Homework**

**Fluency**

1. Which of the following equations would represent the graph of the parabola $y = 3x^2 - 4x - 1$ after a reflection in the $x$-axis?

   (1) $y = -3x^2 - 4x - 1$
   (2) $y = -3x^2 + 4x - 1$
   (3) $y = 3x^2 + 4x - 1$
   (4) $y = -3x^2 + 4x + 1$

2. The graph of $y = 10 - x^2$ represents the graph of $y = x^2$ after

   (1) a vertical shift upwards of 10 units followed by a reflection in the $x$-axis.
   (2) a reflection in the $x$-axis followed by a vertical shift of 10 units upward.
   (3) a leftward shift of 10 units followed by a reflection in the $y$-axis.
   (4) a reflection across the $x$-axis followed by a rightward shift of 10 units.

3. If $f(x) = -2x^2 + 5x - 3$ and $g(x)$ is the reflection of $f(x)$ across the $y$-axis, then an equation of $g$ is which of the following?

   (1) $g(x) = -2x^2 - 5x - 3$
   (2) $g(x) = -2x^2 + 5x + 3$
   (3) $g(x) = 2x^2 + 5x - 3$
   (4) $g(x) = 2x^2 + 5x + 3$

4. If the point $(-3, -5)$ lies on the graph of a function $h(x)$ then which of the following points must lie on the graph of the function $-h(x)$?

   (1) $(3, 5)$
   (2) $(-3, 5)$
   (3) $(-5, -3)$
   (4) $(3, -5)$

5. If the function $y = -f(x - 4)$ were graphed, it would represent which of the following transformations to the graph of $y = f(x)$?

   (1) A rightward shift of 4 units followed by a reflection in the $x$-axis.
   (2) A rightward shift of 4 units followed by a reflection in the $y$-axis.
   (3) A downward shift of 4 units followed by a reflection in the $x$-axis.
   (4) A leftward shift of 4 units followed by a reflection in the $y$-axis.
6. After a reflection in the $x$-axis, the parabola $y = x^2 - 4$ would have the equation

(1) $y = x^2 + 4$  
(2) $y = -x^2 - 4$  
(3) $y = 4 - x^2$  
(4) $y = x^2 - 8$

7. Which of the following equations represents the graph shown below?

(1) $y = (x + 3)^2 + 4$  
(2) $y = -(x + 3)^2 + 4$  
(3) $y = -(x - 3)^2 + 4$  
(4) $y = -(x - 3)^2 - 4$

8. The graph of $f(x) = x^2 + 4x$ is show below on two separate grids. Give an equation and sketch a graph for the functions (a) $f(-x)$ and (b) $-f(x)$.

(a)  
(b)  

**REASONING**

9. If $h(x)$ represents a parabola whose turning point is at $(-3, 7)$ and the function $f$ is defined by $f(x) = -h(x + 2)$, then what are the coordinates of the turning point of $f$? Explain your reasoning.
We have now seen how to shift and reflect functions, specifically in the context of parabolas. In this lesson we will see how to stretch or compress a function in the vertical direction. The first exercise will illustrate this concept with three related parabolas.

**Exercise #1:** Consider the quadratic function \( f(x) = x^2 - 4x - 5 \). The quadratic functions \( g \) and \( h \) are defined by the formulas \( g(x) = 2f(x) \) and \( h(x) = \frac{1}{2}f(x) \).

(a) Determine formulas for both \( g \) and \( h \) in simplest trinomial form.

(b) Using your calculator, sketch and label each curve on the set of axes below. Use the window indicated by the axes.

(c) Using the MINIMUM command on your calculator, determine the minimum value for each function.

\[
f_{\text{min}} = \quad g_{\text{min}} = \quad h_{\text{min}} =
\]

(d) What points did not vary when \( f \) was vertically dilated by factors of 2 and \( \frac{1}{2} \)? Explain why this happened.

---

**VERTICAL DILATIONS OF FUNCTIONS**

The function \( h(x) = k \cdot f(x) \) represents a vertical stretch of the function \( f(x) \) if \( k > 1 \) and a vertical compression of the function \( f(x) \) if \( 0 < k < 1 \).
**Exercise #2:** If the point \((-3, 12)\) lies on the graph of the function \(y = f(x)\), which of the following points must lie on the graph of \(y = 3f(x)\)?

1. \((-9, 36)\)  
2. \((-3, 36)\)  
3. \((-3, 4)\)  
4. \((-9, 12)\)

**Exercise #3:** The graph of \(y = f(x)\) is shown below. Consider the function \(y = g(x)\) defined by \(g(x) = 2f(x) - 3\).

(a) What two transformations have occurred to the graph of \(f\) in order to produce the graph of \(g\)? Specify both the transformations and their order.

(b) Graph and label \(y = g(x)\)

**Exercise #4:** The function \(h(x)\) has a range given by the interval \([2, 10]\). The function \(f(x)\) is defined by \(f(x) = \frac{3}{2}h(x) + 8\). Which of the following gives the range of \(f(x)\)?

1. \([11, 23]\)  
2. \([8, 12]\)  
3. \([15, 27]\)  
4. \([6, 32]\)

**Exercise #5:** If the quadratic function \(g(x)\) has a \(y\)-intercept of 12, which of the following is true about the function \(h(x) = 3g(x) - 5\)?

1. It has a \(y\)-intercept of -5.  
2. It has a \(y\)-intercept of 21.  
3. It has a \(y\)-intercept of -15.  
4. It has a \(y\)-intercept of 31.
VERTICAL STRETCHING AND COMpressing FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLuENCY

1. If the point \((-6, 10)\) lies on the graph of \(y = f(x)\) then which of the following points must lie on the graph of \(y = \frac{1}{2} f(x)\)?

   (1) \((-3, 5)\)  (3) \((-6, 5)\)
   (2) \((-3, 10)\)  (4) \((-12, 20)\)

2. If the function \(h(x)\) is defined as vertical stretch by a factor of 2 followed by a reflection in the \(x\)-axis of the function \(f(x)\) then \(h(x) = \)

   (1) \(2f(-x)\)  (3) \(-\frac{1}{2} f(x)\)
   (2) \(\frac{1}{2} f(x)\)  (4) \(-2f(x)\)

3. If the graph of \(y = x^2\) is compressed by a factor of 3 in the \(y\)-direction and then shifted 4 units down, the resulting graph would have an equation of

   (1) \(y = \frac{1}{3} x^2 - 4\)  (3) \(y = -4x^2 - 3\)
   (2) \(y = -3x^2 - 4\)  (4) \(y = -\frac{1}{3} x^2 + 4\)

4. The quadratic function \(f(x)\) has a turning point at \((-3, 6)\). The quadratic \(y = \frac{2}{3} f(x) + 3\) would have a turning point of

   (1) \((-2, 9)\)  (3) \((-3, 7)\)
   (2) \((1, 7)\)  (4) \((-1, 9)\)

5. The function \(g(x)\) is defined by \(g(x) = -5f(x) + 4\). What three transformations have occurred to the graph of \(f\) to produce the graph of \(g\)? Specify both the transformations and their order.
6. The graph of \( y = h(x) \) is shown below. The function \( f(x) \) is defined by \( f(x) = \frac{-1}{2} h(x) + 3 \).

(a) What three transformations have occurred to the graph of \( h \) to produce the graph of \( f \)? Specify the transformations and the order they occurred in.

(b) Graph and label the function \( f(x) \) on the grid below that contains \( h(x) \).

7. A parabola is shown graphed to the right that is a transformation of \( y = x^2 \). The transformation includes a vertical stretch and a vertical shift. What are the stretch and shift? Based on your answer, write an equation for this parabola.

REASONING

8. The function \( h(x) \) is defined by the equation \( h(x) = 4f(x) - 12 \). Determine two different sets of transformations that could produce the graph of \( h(x) \) from the graph of \( f(x) \). For each, specify two transformations and the order in which they occurred. As a hint, write \( h(x) \) in its factored form.
HORIZONTAL STRETCHING OF FUNCTIONS
COMMON CORE ALGEBRA II

Perhaps one of the hardest transformations of functions occurs when we horizontally stretch and compress the function. Yet these types of transformations can be vital, especially when we model a process over time and then "shrink" the time interval. Let's first explore this using an absolute value function.

Exercise #1: Consider the absolute value function \( f(x) = |x - 2| - 3 \).

(a) Using your calculator, sketch a graph of \( f \) on the axes provided. Label the coordinates of its vertex point without the use of your calculator.

(b) Consider the function \( g(x) = f(2x) \). Determine a formula for \( g \) and then graph it on the axes. Use your calculator to find its minimum point and label it on the graph.

(c) Now consider the function \( h(x) = f\left(\frac{1}{2}x\right) \). Determine a formula for \( h \) and graph it on the axes. Use your calculator to find its minimum point and label it on the graph.

(d) Summarize your findings below for each function.

\[ f(x) \text{ turning point:} \quad f(2x) \text{ turning point:} \quad f\left(\frac{1}{2}x\right) \text{ turning point:} \]

(e) What stayed constant about the turning points? What changed and how did it change?

This one exercise shows a remarkable, and counterintuitive, concept about horizontal dilations:

<table>
<thead>
<tr>
<th>HORIZONTAL DILATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a real number, positive constant such that ( k &gt; 1 ):</td>
</tr>
<tr>
<td>1. The function ( f(kx) ) represents a horizontal compression of ( f(x) ) by a factor of ( k )</td>
</tr>
<tr>
<td>2. The function ( f\left(\frac{1}{k}x\right) ) represents a horizontal stretch of ( f(x) ) by a factor of ( k ).</td>
</tr>
</tbody>
</table>
**Exercise #2:** Let's take a look at the quadratic function \( f(x) = x^2 - 12x + 20 \).

(a) Determine the coordinates of its turning point by using the equation for the axis of symmetry of \( x = -\frac{b}{2a} \).

(b) If \( g \) is defined by \( g(x) = f(3x) \), what should be the coordinates of its turning point based on our previous work? Explain.

(c) Determine a formula for \( g(x) \) and then use the turning point formula to verify your answer from part (b).

(d) Show that the \( y \)-intercept of both \( f(x) \) and \( g(x) \) are equal. What does this make sense from a horizontal dilation perspective?

We can truly investigate why this is happening by looking at a function that is only represented graphically.

**Exercise #3:** Consider the function \( f(x) \) graphed on the grid below. If \( g(x) = f\left(\frac{1}{2}x\right) \) for all values of \( x \) then answer the following questions.

(a) Evaluate each of the following using the definition of \( g \) and then state the point that lies on its graph as a consequence.

\[
\begin{align*}
g(-6) &= \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad g(-2) = \\
g(6) &= \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad g(8) = \\
\end{align*}
\]

(b) Graph \( g \) on the grid to the right. How would you describe its graph compared to the graph of \( f(x) \)?
**Fluency**

1. The quadratic function \( g(x) \) has a turning point at \((-12, 8)\). Where would the quadratic function \( f(x) = g(4x) \) have a turning point?

   (1) \((-48, 32)\)  
   (2) \((-48, 8)\)  
   (3) \((-3, 8)\)  
   (4) \((-3, 2)\)

2. The three exponential graphs shown below represent the function \( f(x) = b^x \), \( g(x) = b^{2x} \) and \( h(x) = b^{\frac{x}{2}} \), for some \( b > 1 \).

   (a) Label each with its correct equation.

   (b) Algebraically, show that the \( y \)-intercept of each function is the same.

3. The graph of \( f(x) \) is shown on the grid below. Sketch a graph of \( f(2x) \) on the same set of axes.

State the domain of the two functions:

Domain of \( f(x) \):  
Domain of \( f(2x) \):
APPLICATIONS

4. An arch is to be constructed so that its shape follows the curve \( y = -\frac{1}{2}x^2 + 10x \), where \( x \) measures the horizontal distance along the ground and \( y \) measures the vertical height of the arch above the ground, both in units of feet. The general graph of this arch is shown below.

(a) Based on this equation, what is the height of the arch at the turning point? Show the work that leads to your answer.

(b) If a second arch was to be created that had the same height, but only half the width, determine an equation for this arch based on our work in this lesson.

(c) Choosing an appropriate graphing window based on (a), graph the second arch on the axes above. Label your graphing window. Use your calculator to determine the new turning point and label both points on the graphs.

REASONING

5. We've seen repeatedly that a horizontal dilation does not alter the graph's \( y \)-intercept. Given the function \( f(x) \) and \( g(x) = f(kx) \), can you determine an algebraic argument for why \( f(x) \) and \( g(x) \) must have the same \( y \)-intercepts? (Hint: Think about how we always find the \( y \)-intercept of any function).

6. If the function \( f(x) \) has a domain of \(-2 \leq x \leq 8\) and a range of \(-4 \leq y \leq 6\) and the function \( g(x) \) is defined by the formula \( g(x) = 5f(2x) \) then what are the domain and range of \( g \)? Explain your thought process.
Recall that functions are simply rules that convert inputs into outputs. These rules then get placed into various
categories, such as linear functions, exponential functions, quadratic functions, etcetera, based on shared
characteristics. In this lesson you will learn another way to classify some functions that have useful
symmetries.

**EVEN AND ODD FUNCTIONS**

A function is known as **even** if \( f(-x) = f(x) \) for every value of \( x \) in the domain of \( f(x) \).

A function is known as **odd** if \( f(-x) = -f(x) \) every value of \( x \) in the domain of \( f(x) \).

**Exercise #1:** It's good to be able to "read" mathematical definitions and theorems for yourself as you advance in
your mathematical studies. Look at the definitions above and try to determine what they say about the inputs
and outputs for these types of functions. Highlight or underline words you find important. Then, write down
your interpretation on the lines.

1. Even Functions: _____________________________________________

2. Odd Functions: ____________________________________________

Let's take a look at **even** and **odd** functions first from a graphical standpoint.

**Exercise #2:** Consider the partial graph of the function \( f(x) \) shown twice below. Sketch the other half of the
function if in (a) \( f(x) \) is **even** and in (b) \( f(x) \) is **odd**. The three coordinate pairs are listed to help you plot.

(a) **even**

(b) **odd**

(c) Describe the symmetry of the **even** graph and the **odd** graph. Use as technically correct terminology as you
can from your studies in Geometry.
Some of the functions you have seen in your studies so far are even, some have been odd, and many have been neither. Let's take a look at a variety of functions and consider whether they fall into one of these categories.

**Exercise #3:** Consider the function $f(x) = |x| - 4$.

(a) Evaluate this function for a variety of opposite input pairs. What type (even, odd, or neither) does $f$ appear to be?

(b) Sketch $f(x)$ on the grid below *without* the use of your calculator. Does it have the correct symmetry for your choice in (a)?

We will be studying higher-order polynomials extensively in Unit #10, but we can take a look at a simple **cubic polynomial**.

**Exercise #4:** Let's investigate $g(x) = x^3 - 4x$.

(a) Use your calculator's table option to fill in the following table. What type of function is this. Explain.

(b) Sketch a graph of $g(x)$ using your calculator and the window indicated.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

**Exercise #5:** Is the simple exponential function $f(x) = 2^x$ odd, even, or neither? Support your argument with numerical evidence.
EVEN AND ODD FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Given the partially filled out table below for \( f(x) \), fill out the rest of it based on the function type.

   (a) Even
   
   \[
   \begin{array}{c|cccccc}
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   \hline
   y & 5 & -7 & 4 & -4 & & & \\
   \end{array}
   \]

   (b) Odd
   
   \[
   \begin{array}{c|cccccc}
   x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
   \hline
   y & 5 & -7 & 0 & -4 & & & \\
   \end{array}
   \]

2. Half of the graph of \( f(x) \) is shown below. Sketch the other half based on the function type.

   (a) Even
   
   ![Graph of an even function]

   (b) Odd
   
   ![Graph of an odd function]

3. If \( f(x) \) is an even function and \( f(3) = 5 \) then what is the value of \( 4f(3) + 2f(-3) \)?

   (1) 30
   (2) 60
   (3) 10
   (4) 6

4. If \( g(x) \) is an odd, one-to-one function and if \( g(7) = -2 \), then which of the following points must lie on the graph of the inverse of \( g(x) \), \( g^{-1}(x) \). Explain how you made your choice.

   (1) \((-7, 2)\)
   (2) \((2, -7)\)
   (3) \((2, 7)\)
   (4) \((7, -2)\)
5. Which of the following functions is even? Explain how you arrived at your choice.

(1) \( y = x^2 - 4x \)  
(2) \( y = |x - 6| \)  
(3) \( y = 9 - x^2 \)  
(4) \( y = 4^x \)

6. The function \( f(x) = \frac{4x^2 + 2}{x} \) is either even or odd. Determine which by exploring the function using tables on your calculator. Provide evidence for your final choice.

7. Generally, logarithms are not defined for negative inputs. This obstacle can be overcome by composing a logarithm function with an absolute value function. Consider the function \( f(x) = \log_2 |x| \).

(a) If the graph of \( y = \log_2 (x) \) is shown below, sketch the other half of \( f \).

(b) What type of function is \( f(x) \)?

**REASONING**

8. You may have noticed that every odd function that we drew that was defined at \( x = 0 \) passed through the origin, \((0, 0)\). Why must this always be true?

9. Even functions have symmetry across the \( y \)-axis. Odd functions have symmetry across the origin. Can a function have symmetry across the \( x \)-axis? Why or why not?
UNIT #8

RADICALS AND THE QUADRATIC FORMULA

Lesson #1 – Square Root Functions
Lesson #2 – Solving Square Root Equations
Lesson #3 – The Basic Exponent Properties
Lesson #4 – More Work with Fractional Exponents
Lesson #5 – More Exponent Practice
Lesson #6 – The Quadratic Formula
Lesson #7 – More Work with the Quadratic Formula
Square roots are the natural inverses of squaring. In other words, to find the square root of an input, we must find a number that when squared gives the input. Because of their important role in higher-level mathematics, it is important to understand their graphs, as well as their domains and ranges. In this lesson we will explore all of these facets of this common function.

**Exercise #1:** Consider the two functions \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{x+3} - 2 \).

(a) Graph \( y = f(x) \) without the use of your calculator on the grid shown. Label its equation.

(b) Using your calculator to generate a table of values, graph \( y = g(x) \) on the same grid and label its equation. Start your table at \( x = -10 \) to see certain \( x \)-values not in the domain of this function.

(c) State the domain and range of each function below using set-builder notation.

\[
\begin{align*}
    f(x) &= \sqrt{x} \\
    g(x) &= \sqrt{x+3} - 2
\end{align*}
\]

Domain: Domain:

Range: Range:

**Exercise #2:** Which of the following equations would represent the graph shown below?

(1) \( y = -\sqrt{x+4} \)

(2) \( y = 4 - \sqrt{x} \)

(3) \( y = \sqrt{x-4} \)

(4) \( y = -\sqrt{x-4} \)
As we saw in the first exercise, the domains of square root functions are oftentimes limited due to the fact that square roots of negative numbers do not exist in the Real Number System. We shall see in Unit #9 how these square roots can be defined if a new type of number is introduced. For now, though, we are only working with real numbers.

**Exercise #3:** Which of the following values of $x$ does *not* lie in the domain of the function $y = \sqrt{x - 5}$? Explain why it does not lie there.

1. $x = 6$
2. $x = 2$
3. $x = 5$
4. $x = 7$

**Exercise #4:** Determine the domain for each of the following square root functions. Show an inequality that justifies your work.

(a) $y = \sqrt{x + 2}$
(b) $y = \sqrt{3x - 2}$
(c) $y = \sqrt{8 - 2x}$

**Exercise #5:** Consider the function $f(x) = \sqrt{x^2 + 4x - 12}$.

(a) Use your calculator to sketch the function on the axes given.

(b) Set up and solve a quadratic inequality that yields the domain of $f(x)$. 
**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

1. Which of the following represents the domain and range of \( y = \sqrt{x - 5} + 7 \)? Solve this either by considering the shifting that has occurred to \( y = \sqrt{x} \) or by producing a graph on your calculator.

   (1) Domain: \([-5, \infty)\)  Range: \([7, \infty)\)
   (3) Domain: \((-7, \infty)\)  Range: \((5, \infty)\)

   (2) Domain: \([5, \infty)\)  Range: \([7, \infty)\)
   (4) Domain: \([7, \infty)\)  Range: \([5, \infty)\)

2. Which of the following values of \( x \) is not in the domain of \( y = \sqrt{1 - 3x} \)?

   (1) \( x = \frac{1}{3} \)
   (3) \( x = 0 \)

   (2) \( x = -1 \)
   (4) \( x = 4 \)

3. Which of the following equations describes the graph shown below?

   (1) \( y = \sqrt{x+4} + 1 \)
   (2) \( y = \sqrt{x-4} - 1 \)
   (3) \( y = \sqrt{x+4} - 1 \)
   (4) \( y = \sqrt{x-4} + 1 \)

4. Which equation below represents the graph shown?

   (1) \( y = \sqrt{x-2} - 5 \)
   (2) \( y = -\sqrt{x+2} + 5 \)
   (3) \( y = -\sqrt{x-2} + 5 \)
   (4) \( y = \sqrt{x+2} + 5 \)
5. Determine the domains of each of the following functions. State your answers in set-builder notation.

(a) \( y = \sqrt{x+10} \)  
(b) \( y = \sqrt{3x-5} \)  
(c) \( y = \sqrt{7-2x} \)

6. Set up and algebraically solve a quadratic inequality that results in the domain of each of the following. Verify your answers by graphing the function in a standard viewing window.

(a) \( y = \sqrt{x^2 - 4x - 5} \)  
(b) \( y = \sqrt{9-x^2} \)

7. Consider the function \( g(x) = -\sqrt{x+5} + 3 \).

(a) Graph the function \( y = g(x) \) on the grid shown.

(b) Describe the transformations that have occurred to the graph of \( y = \sqrt{x} \) to produce the graph of \( y = g(x) \). Specify both the transformations and their order.
Equations involving square roots arise in a variety of contexts, both applied and purely mathematical. As always, the key to solving these equations lies in the applications of inverse operations. The key inverse relationship in these equations is that between taking a square root and squaring.

**Exercise #1:** Solve each of the following square root equations, which are arranged from less to more complex. Check each equation.

(a) \( \sqrt{x} = 7 \)  
(b) \( \sqrt{x - 3} = 5 \)  
(c) \( \sqrt{2x - 1} = 4 \)

(d) \( 3\sqrt{x} - 4 = 20 \)  
(e) \( 2\sqrt{x} + 5 + 7 = 13 \)  
(f) \( 5\sqrt{3x - 2} - 4 = 36 \)

**Exercise #2:** Which of the following is the solution to \( 3\sqrt{\frac{x}{2}} = 15 \)?

(1) \( x = 12.5 \)  
(2) \( x = 25 \)

(3) \( x = 50 \)  
(4) \( x = 4050 \)
A more complicated scenario arises when a square root expression is equal to a linear expression. The next exercise will illustrate both the graphical and algebraic issues involved.

**Exercise #3:** Consider the system of equations shown below.

\[ y = \sqrt{x+3} \quad \text{and} \quad y = x + 1 \]

(a) Solve this system graphically using the grid to the right.

(b) Solve this system *algebraically* for only the \( x \)-values using substitution below.

(c) Why does your answer from part (a) contradict what you found in part (b)?

Oftentimes, roots are introduced by various algebraic techniques that for one reason or another are not valid solutions of the equations. These roots are known as *extraneous* and can always be found by checking within the original equation.

**Exercise #4:** Find the solution set of each of the following equations. Be sure to check your work and reject any extraneous roots.

(a) \( \sqrt{2x-3} = x-3 \)  

(b) \( 2x = \sqrt{x+6} - 2 \)
SOLVING SQUARE ROOT EQUATIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following square root equations. As in the lesson, they are arranged from lesser to more complex. Check your answers.

(a) \( \sqrt{x} = 5 \)  
(b) \( \sqrt{x + 2} = 10 \)  
(c) \( \sqrt{\frac{2x}{3}} = 6 \)

(d) \( 4\sqrt{x} = 24 \)  
(e) \( 2\sqrt{x} = 1 \)  
(f) \( \sqrt{3x + 4} = 8 \)

(g) \( \frac{1}{2}\sqrt{x} - 5 = 2 \)  
(h) \( \sqrt{4x - 1} + 3 = 4 \)  
(i) \( 5\sqrt{1 - 5x} - 3 = 27 \)

(j) \( \sqrt{x^2 - 10x + 25} = 5 \)  
(k) \( \sqrt{2x^2 + 17x} = 3 \)  
(l) \( \sqrt{3x^2 + 7x + 10} = 4 \)
2. Which of the following values solves the equation $\frac{\sqrt{4x+19}}{2} = 2$?

(1) $-\frac{9}{2}$  (3) $\frac{4}{3}$
(2) $-\frac{3}{4}$  (4) $\frac{1}{2}$

3. Solve each of the following equations for all values of $x$. Check your possible solutions in the original equation. Reject any extraneous roots.

(a) $x - 1 = \sqrt{x + 11}$

(b) $\sqrt{4x + 36} = 2x - 6$

4. Solve each of the following equations for all values of $x$. As in problem #1, be sure to isolate the square root expression first before squaring both sides of the equation. Check your possible solutions in the original equation. Reject any extraneous roots.

(a) $6x = 2\sqrt{24x + 17} - 8$

(d) $\frac{\sqrt{6x + 4} - 1}{4} = x$
**THE BASIC EXPONENT PROPERTIES**

**COMMON CORE ALGEBRA II**

Exponents, which indicate repeated multiplication, are extremely important in higher-level mathematical study because of their importance in numerous areas. The rules they play by, known as the exponent properties, are critical to master. We will develop each one of the major seven properties. The order in which these are presented is not unique.

**Exercise #1 (Property #1):** Consider the expression $x^a \cdot x^b$. How can we rewrite this product equivalently?

(a) Rewrite $x^2 \cdot x^4$. Write as an extended product first if necessary.  
(b) Generalize: $x^a \cdot x^b =$

(c) Practice - Rewrite each of the following in simplest form:

(i) $x^{10} \cdot x^3$  
(ii) $(5x^4)(6x^3)$  
(iii) $x^3 y^3 x^6 y$

**Exercise #2 (Property #2):** Consider the expression $\frac{x^a}{x^b}$. How can we simplify this quotient (division)?

(a) Rewrite the quotient $\frac{x^5}{x^2}$ in simplest form by using the Property #1 and the multiplication property of fractions.  
(b) Generalize for $a > b$: $\frac{x^a}{x^b} =$

(c) Practice - Rewrite each of the following in simplest form:

(i) $\frac{x^8}{x^2}$  
(ii) $\frac{6x^{10}}{12x^3}$  
(iii) $\frac{x^6 y^3}{xy^2}$

**Exercise #3 (Property #3):** But what if the power of the numerator is less than that of the denominator?

(a) Rewrite the quotient $\frac{x^2}{x^5}$ as in Exercise #2(a).  
(b) What results if you apply Exponent Prop #2?

(c) Generalize: $x^{-a} =$  
(d) Rewrite each of the following without the use of negative exponents:

(i) $2^{-3} =$  
(ii) $x^{-4} =$
**Exercise #4 (Property #4):** What if the powers are the same?

(a) Rewrite the quotient \( \frac{x^3}{x^3} \) based on the fundamental concept of dividing a quantity by itself.

(b) What results if you apply Exponent Prop #2?

(c) Generalize: \( x^0 = \)

(d) Simplify each of the following:

\( (i) \, 5^0 = \quad (ii) \, 3x^0 = \)

**Exercise #5 (Property #5):** Now let's take a look at the very common scenario of \( (x^a)^b \).

(a) Rewrite \( (x^2)^3 \). Write as an extended product first if necessary.

(b) Generalize: \( (x^a)^b = \)

(c) Which of the following expressions is *not* equivalent to \( x^{30} \)?

1. \( (x^{10})^3 \)
2. \( (x^6)^5 \)
3. \( x^5 \cdot x^6 \)
4. \( x^{10} \cdot x^{20} \)

The final two properties we will look at concern how exponents **distribute** over multiplication and division.

**Exercise #6 (Properties #6 and #7):** Let's take a look at \( (xy)^a \).

(a) Rewrite \( (xy)^3 \) using the definition of an exponent along with the associative and commutative properties of multiplication.

(b) Generalize: \( (xy)^a = \)

(c) Rewrite \( \left( \frac{x}{y} \right)^3 \) using the definition of an exponent along with the multiplication property of fractions.

(d) Generalize: \( \left( \frac{x}{y} \right)^a = \)

(e) Rewrite each of the following as equivalent expressions:

\( (i) \, (2x^2)^3 \quad (ii) \, \left( \frac{3}{x^3} \right)^2 \quad (iii) \, \left( \frac{-2x^2y^5}{3z^3} \right)^3 \)
THE BASIC EXPONENT PROPERTIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Express each of the following expressions in "expanded" form, i.e., do all of the multiplication and/or division possible and combine as many exponents as possible.

   (a) $x^3 \cdot x^{12}$
   (b) $4x^3 \cdot 5x^5$
   (c) $(-3x^2y)(5x^7y^3)$
   (d) $(4x^3y^6)(-7x^4)$

   (e) $\frac{x^9}{x^3}$
   (f) $\frac{5x^3y^7}{15xy^2}$
   (g) $\frac{x^3}{x^{10}}$
   (h) $\frac{10x^4y^3}{25x^8}$

   (i) $(x^5)^8$
   (j) $(10x^3)^0$
   (k) $(-4x^2)^3$
   (l) $(x^{-2})^4$

2. Which of the following is not equal to $2^{-2}$? Do not use your calculator to do this problem.

   (1) $\frac{1}{4}$
   (2) $-4$
   (3) $0.25$
   (4) $\frac{1}{2^2}$

3. If the expression $\frac{1}{2x}$ was placed in the form $ax^b$ where $a$ and $b$ are real numbers, then which of the following is equal to $a + b$? Show how you arrived at your answer.

   (1) 1
   (2) $\frac{3}{2}$
   (3) $\frac{1}{2}$
   (4) $-\frac{1}{2}$
4. If \( f(x) = 5x^0 + 4x^{-3} \) then \( f(a) = \)

(1) \( 12a - 5 \) \hspace{1cm} (3) \( \frac{1}{4a^3} + 5 \)

(2) \( \frac{5 + 4}{a^7} \) \hspace{1cm} (4) \( -12a + 1 \)

5. Which of the following is equivalent to \( \frac{(4x^8)^3}{(6x^5)^2} \) for all \( x \neq 0 \)? Show the manipulations that lead to your final answer.

(1) \( \frac{16}{9} x^{14} \) \hspace{1cm} (3) \( \frac{2}{3} x^{14} \)

(2) \( \frac{16}{9} x^4 \) \hspace{1cm} (4) \( \frac{2}{3} x^4 \)

APPLICATIONS

6. It is helpful to be able to think about very large numbers in terms of powers of 10. You should be familiar with many of these terms, but have you thought about how many 10’s are multiplying each other? Here are some numbers to think about and examples of things that would be counted in these quantities. Fill in the proper power of 10. The first has been done for you.

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>POWER OF 10</th>
<th>EXAMPLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 million</td>
<td>( = 1,000 \cdot 1,000 = 10^3 \cdot 10^3 = 10^6 )</td>
<td>The distance between New York City and Boston is approximately 1 million feet.</td>
</tr>
<tr>
<td>1 billion</td>
<td>( = 1,000 \cdot 1 \text{ million} = 10^3 \cdot 10^6 = )</td>
<td>There are approximately 3 billion seconds in a century.</td>
</tr>
<tr>
<td>1 trillion</td>
<td>( = (1 \text{ million})^2 = (10^6)^2 = )</td>
<td>There are 6 trillion miles in a light year, i.e. the distance light can travel in a year.</td>
</tr>
<tr>
<td>1 quadrillion</td>
<td>( = 1000 \cdot 1 \text{ trillion} = )</td>
<td>There are approximately 1 quadrillion ants populating the earth at any time.</td>
</tr>
<tr>
<td>1 quintillion</td>
<td>( = (1 \text{ billion})^2 = )</td>
<td>There are approximately 8 quintillion grains of sand on all of the Earth's beaches.</td>
</tr>
</tbody>
</table>

REASONING

7. The functions \( f(x) = 2^x \) and \( g(x) = 8(2)^x \) are both shown graphed. The graph of \( g \) is certainly a vertical stretch of the function \( f \) by a factor of 8. But, it is also a shift of \( f \) by three units left? Can you explain why this is using an exponent law?
Recall that in Unit #4 we introduced the concept that roots (square roots, cube roots, etcetera) could be represented by rational or fractional exponents.

**UNIT FRACTION EXPONENTS**

For $n$ given as a positive integer: 

$$b^{1/n} = \sqrt[n]{b}$$

**Exercise #1:** Rewrite each expression in the form $ax^b$ where $a$ and $b$ are both rational numbers.

(a) $5\sqrt{x}$  
(b) $\frac{3\sqrt{x}}{4}$  
(c) $\frac{7}{3\sqrt{x}}$  
(d) $\frac{5}{3\sqrt[3]{x}}$

Recall we can also combine integer powers with roots with the following:

**RATIONAL EXPONENT CONNECTION TO ROOTS**

For the rational number $\frac{m}{n}$, $b^{m/n}$ is equivalent to: $\sqrt[n]{b^m}$ or $(\sqrt[n]{b})^m$.

**Exercise #2:** Rewrite each of the following power/root combinations as a rational exponent in simplest form.

(a) $\sqrt{x^7}$  
(b) $\frac{3\sqrt{x^6}}{}$  
(c) $(\sqrt{x})^6$  
(d) $(\frac{3\sqrt{x}}{})^{10}$

**Exercise #3:** If $f(x) = 10x^{3/2} - 24x^{-1}$, then which of the following represents the value of $f(4)$? Find the value without the use of a calculator. Show the steps in your calculation.

(1) 36  
(2) 48  
(3) 54  
(4) 74

**Exercise #4:** Which of the following is not equivalent to $x^{-7/3}$?

(1) $\frac{1}{\sqrt[3]{x^7}}$  
(2) $\frac{1}{\sqrt{x^3}}$  
(3) $\frac{1}{\sqrt[3]{x^7}}$  
(4) $\frac{1}{\sqrt[7]{x^3}}$
Fractional exponents play by the same rules (properties) as all other exponents. It is, in fact, these properties that can justify many standard manipulations with square roots (and others). For example, simplifying roots.

**Exercise #5:** We only consider a square root "simplified" when all of its **perfect square factors** have had their square roots evaluated.

(a) Fill in the exponent property below:

\[(ab)^n = \]

(b) Rewrite \( \sqrt{28} \) in factored form, with one factor being the largest perfect square divisor. Also, write the square root in exponent form.

(c) Simplify \( \sqrt{28} \) using (b) and the property from (a).

(d) Generalize: \( \sqrt{a \cdot b} = \)

\[ a \cdot b = \]

**Exercise #6:** Simplify each of the following square roots. Show the manipulations that lead to your answers.

(a) \( \sqrt{18x^4} \)  
(b) \( \sqrt{200x^5y^3} \)  
(c) \( \sqrt{147x^9y^4} \)

We can extend the simplifying process to include cube roots and higher-order roots by simply extending our thinking.

**Exercise #7:** Simplify each of the following higher order roots.

(a) \( \sqrt[3]{16} \)  
(b) \( \sqrt[3]{108} \)  
(c) \( \sqrt[3]{250} \)  
(d) \( \sqrt[3]{128x^8} \)

(e) \( \sqrt[3]{162} \)  
(f) \( \sqrt[3]{16x^8} \)  
(g) \( \sqrt[3]{48x^{10}y^5} \)  
(h) \( \sqrt[3]{64x^{12}y^{15}} \)
FRACTIONAL EXPONENTS REVISITED
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is equivalent to \( x^{5/2} \)?
   
   (1) \( \frac{5x}{2} \)  
   (2) \( \frac{2x}{5} \)
   (3) \( \sqrt{x^5} \)
   (4) \( \sqrt[5]{x^2} \)

2. If the expression \( \frac{1}{\sqrt{x}} \) was placed in \( x^a \) form, then which of the following would be the value of \( a \)?
   
   (1) \(-2\)  
   (2) \(2\)  
   (3) \(\frac{1}{2}\)  
   (4) \(-\frac{1}{2}\)

3. Which of the following is not equivalent to \( \sqrt{x^9} \)?
   
   (1) \(x^3\)  
   (2) \(\left(\sqrt{x}\right)^9\)  
   (3) \(x^{9/2}\)  
   (4) \(x^4\sqrt{x}\)

4. The radical expression \( \sqrt{50x^5y^3} \) can be rewritten equivalently as
   
   (1) \(25xy\sqrt{2xy}\)  
   (2) \(5xy\sqrt{xy}\)  
   (3) \(5x^2y\sqrt{2xy}\)  
   (4) \(10x^2y\sqrt{5xy}\)

5. If the function \( y = 12\sqrt{x} \) was placed in the form \( y = ax^b \) then which of the following is the value of \( a \cdot b \)?
   
   (1) \(-36\)  
   (2) \(-4\)  
   (3) \(36\)  
   (4) \(4\)
6. Rewrite each of the following expressions without roots by using fractional exponents.

(a) $\sqrt{x}$  
(b) $\sqrt[3]{x}$  
(c) $\sqrt[4]{x}$  
(d) $\sqrt[5]{x^3}$

(e) $\sqrt[3]{x^{11}}$  
(f) $\frac{1}{\sqrt[4]{x}}$  
(g) $\frac{1}{\sqrt[5]{x^2}}$  
(h) $\frac{1}{\sqrt[5]{x^9}}$

7. Rewrite each of the following without the use of fractional or negative exponents by using radicals.

(a) $x^{\frac{1}{6}}$  
(b) $x^{\frac{1}{10}}$  
(c) $x^{-\frac{1}{3}}$  
(d) $x^{-\frac{1}{3}}$

(e) $x^{\frac{7}{2}}$  
(f) $x^{-\frac{7}{2}}$  
(g) $x^{\frac{9}{4}}$  
(h) $x^{-\frac{7}{11}}$

8. Simplify each of the following square roots that contain variables in the radicand.

(a) $\sqrt{8x^9}$  
(b) $\sqrt{75x^{16}y^{11}}$  
(c) $2x\sqrt{18x^7}$  
(d) $3x^2y\sqrt{98x^5y^8}$

9. Express each of the following roots in simplest radical form.

(a) $\sqrt[3]{16x^8}$  
(b) $\sqrt[3]{108x^5y^{10}}$  
(c) $\sqrt[3]{64x^{12}y^{14}}$  
(d) $\sqrt[3]{375x^7y^{11}}$

10. Mikayla was trying to rewrite the expression $25x^{\frac{1}{2}}$ in an equivalent form that is more convenient to use. She incorrectly rewrote it as $5\sqrt{x}$. Explain Mikalya's error.
For further study in mathematics, especially Calculus, it is important to be able to manipulate expressions involving exponents, whether those exponents are positive, negative, or fractional. The basic laws of exponents, which you should have learned in Algebra 1 and have used previously in this course, are shown to the right. They apply regardless of the nature of the exponent (i.e. positive, negative, or fractional).

Although these problems can be challenging, the key will be to carefully apply these exponent laws in a systematic manner.

Exercise #1: Simplify each of the following expressions. Leave no negative exponents in your answers.

(a) \( \frac{x^3 \cdot x^4}{(x^5)^2} \)  
(b) \( \frac{(x^2 y)^4}{x^5 y^7} \)  
(c) \( \frac{x^2 y^4}{x^6 y} \)  
(d) \( \frac{(x^3 y^{-4})^2}{(xy^3)^4} \)

In the last exercise, all of the powers were integers. In the next exercise, we introduce fractional powers. Remember, though, that they will still follow the exponent rules above. If needed, use your calculator to help add and subtract the powers.

Exercise #2: Simplify each of the following expressions. Write each without the use of negative exponents.

(a) \( \frac{x^{1/3} \cdot x^{1/2}}{x^{1/6}} \)  
(b) \( \frac{(x^{1/2})^5}{x^{2/3} \cdot x^3} \)  
(c) \( \frac{4x^{2/3}^3}{32x^8} \)
We must not forget from our last lesson that fractional exponents have an equivalent interpretation as roots. We should be able to move from one representation to another.

**Exercise #3:** Rewrite each expression below in both its simplest form and using radical expressions.

(a) \( x^{\frac{2}{3}} \)  
(b) \( \frac{x^{\frac{3}{2}}}{x^{\frac{2}{3}}} \)  
(c) \( \frac{1}{x^{-\frac{1}{2}}} \)

(d) \( \frac{x^3}{\sqrt{x}} \)  
(e) \((8x^3)^{\frac{2}{3}}\)  
(f) \(\frac{(27x)^{\frac{1}{3}}}{6\sqrt{x}}\)

**Exercise #4:** Which of the following is equivalent to \( \sqrt[3]{8x^7} \)?

(1) \( 8x^{\frac{7}{3}} \)  
(2) \( 2x^{\frac{7}{3}} \)  
(3) \( 2x^{\frac{7}{3}} \)  
(4) \( 8x^{\frac{7}{3}} \)

**Exercise #5:** The expression \( \frac{1}{\sqrt[4]{4x}} \) is the same as

(1) \( \frac{1}{2}x^{-\frac{1}{2}} \)  
(2) \( 2x^{\frac{1}{2}} \)  
(3) \( 4x^{\frac{1}{2}} \)  
(4) \( \frac{1}{2}x^{\frac{1}{2}} \)
EXPONENT PRACTICE
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Rewrite each of the following expressions in simplest form and without negative exponents.

   (a) \( \frac{x^3 x^7}{(x^2)^3} \)  

   (b) \( \frac{5x^4}{25x^{10}} \)  

   (c) \( \frac{(x^3 y^4)^2}{(x^3 y)^3} \)  

   (d) \( \frac{(2x^3)^5}{8x^{-3}} \)

2. Which of the following represents the value of \( \frac{a^4}{b^2} \) when \( a = 3 \) and \( b = 2 \)?

   (1) \( \frac{4}{9} \)  

   (2) \( \frac{4}{81} \)  

   (3) \( \frac{1}{36} \)  

   (4) \( \frac{1}{3} \)

3. Simplify each expression below so that it contains no negative exponents. Do not write the expressions using radicals.

   (a) \( \frac{x^{7/2} y^{1/2}}{3^{4/2} y^{2}} \)  

   (b) \( \frac{(x^{1/3})^4}{x^{-2/3}} \)  

   (c) \( \left(5x^{3/2} y^{-1/2}\right) \left(2x^2 y^{-3}\right) \)

4. Which of the following represents the expression \( \frac{24x^{-7/2}}{6x^{5/2}} \) written in simplest form?

   (1) \( \frac{4}{x^3} \)  

   (2) \( 4x^3 \)  

   (3) \( \frac{x^2}{4} \)  

   (4) \( 4x^2 \)
5. Rewrite each of the following expressions using radicals. Express your answers in simplest form.

(a) \((4x)^{\frac{3}{2}}\)  
(b) \(x^{-\frac{2}{3}}\)  
(c) \((x^4)^{\frac{3}{5}}\)

(d) \(\sqrt[3]{x} \div \sqrt{x}\)  
(e) \(\frac{\sqrt{x} \cdot x^2}{x^{\frac{5}{3}}}\)  
(f) \(\frac{(2\sqrt{x})^3}{24x}\)

6. Which of the following is equivalent to \(\frac{5\sqrt{x}}{20x^3}\)?

(1) \(\frac{1}{4\sqrt{x^3}}\)  
(2) \(\frac{4}{\sqrt{x^5}}\)  
(3) \(\frac{1}{4\sqrt{x^2}}\)  
(4) \(\frac{1}{4\sqrt{x^5}}\)

7. When written in terms of a fractional exponent the expression \(\frac{\sqrt{x} \cdot x}{x^{-2}}\) is

(1) \(x^{\frac{3}{2}}\)  
(2) \(x^{\frac{5}{2}}\)  
(3) \(x^{-\frac{1}{2}}\)  
(4) \(x^{-\frac{3}{2}}\)

8. Expressed as a radical expression, the fraction \(\frac{x^{\frac{1}{3}} \cdot x^{\frac{1}{2}}}{x^{-1}}\) is

(1) \(\frac{1}{\sqrt[6]{x}}\)  
(2) \(\frac{1}{\sqrt[10]{x^6}}\)  
(3) \(\sqrt[6]{x^6}\)  
(4) \(\sqrt[10]{x^{11}}\)
THE QUADRATIC FORMULA
COMMON CORE ALGEBRA II

We saw in Common Core Algebra I how any quadratic equation could be solved using the process of Completing the Square. This is reviewed in Exercise #1.

**Exercise #1:** Solve the following quadratic equation for all values of \( x \) by first completing the square on the quadratic expression. Express your answers in simplest radical form.

\[ x^2 - 6x + 1 = 0 \]

Since any quadratic can be rearranged through the process of Completing the Square, a formula can be developed that will solve for the roots of any quadratic equation. This famous formula, known as the Quadratic Formula, is shown below. You worked with this as well in Algebra I.

**THE QUADRATIC FORMULA**

The solutions to the quadratic equation \( ax^2 + bx + c = 0 \), assuming \( a \neq 0 \), are given by

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**Exercise #2:** Using the quadratic formula shown above, solve the equation from Exercise #1. State your answers in simplest radical form.

**Exercise #3:** Which of the following represents the solutions to the equation \( x^2 - 10x + 20 = 0 \)?

(1) \( x = 5 \pm \sqrt{10} \)  
(2) \( x = -5 \pm \sqrt{5} \)  
(3) \( x = -5 \pm \sqrt{10} \)  
(4) \( x = 5 \pm \sqrt{5} \)
Although the quadratic formula is most helpful when a quadratic expression is **prime** (not factorable), it can be used as a replacement for the Zero Product Law in cases where the quadratic can be factored.

**Exercise #4:** Solve the quadratic equation shown below in two different ways – (a) by factoring and (b) by using the quadratic formula.

(a) \(2x^2 + 11x - 6 = 0\) by factoring  
(b) \(2x^2 + 11x - 6 = 0\) by the quadratic formula

The quadratic formula is very useful in algebra - it should be committed to memory with practice.

**Exercise #5:** Solve each of the following quadratic equations by using the quadratic formula. Some answers will be purely rational numbers and some will involve irrational numbers. Place all answers in simplest form.

(a) \(3x^2 + 5x + 2 = 0\)  
(b) \(x^2 - 8x + 13 = 0\)

(c) \(2x^2 - 2x - 5 = 0\)  
(d) \(5x^2 + 8x - 4 = 0\)
THE QUADRATIC FORMULA
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following quadratic equations using the quadratic formula. Express all answers in simplest form.

(a) $x^2 + 7x - 18 = 0$       (b) $x^2 - 2x - 1 = 0$

(c) $x^2 + 8x + 13 = 0$       (d) $3x^2 - 2x - 3 = 0$

(e) $6x^2 - 7x + 2 = 0$       (f) $5x^2 + 3x - 4 = 0$
2. Which of the following represents all solutions of \( x^2 - 4x - 1 = 0 \)?

   (1) \( 2 \pm \sqrt{5} \)  
   (2) \( -2 \pm \sqrt{5} \)  
   (3) \( 2 \pm \sqrt{10} \)  
   (4) \( -2 \pm \sqrt{12} \)

3. Which of the following is the solution set of the equation \( 4x^2 - 12x - 19 = 0 \)?

   (1) \( \frac{5}{2} \pm \sqrt{3} \)  
   (2) \( -\frac{2}{3} \pm \sqrt{2} \)  
   (3) \( \frac{3}{2} \pm \sqrt{7} \)  
   (4) \( -\frac{7}{3} \pm \sqrt{6} \)

4. Rounded to the nearest hundredth the larger root of \( x^2 - 22x + 108 = 0 \) is

   (1) 18.21  
   (2) 13.25  
   (3) 6.74  
   (4) 14.61

5. Algebraically find the \( x \)-intercepts of the quadratic function whose equation is \( y = x^2 - 4x - 6 \). Express your answers in simplest radical form.

Applications

6. A missile is fired such that its height above the ground is given by \( h = -9.8t^2 + 38.2t + 6.5 \), where \( t \) represents the number of seconds since the rocket was fired. Using the quadratic formula, determine, to the nearest tenth of a second, when the rocket will hit the ground.
In the last lesson we learned and practiced use of the quadratic formula (shown above). This formula is extremely useful because it allows us to solve quadratic equations, whether they are prime or factorable. In this lesson, we will get more practice using this formula.

**Exercise #1:** Consider the quadratic function \( f(x) = x^2 - 4x - 36 \).

(a) Algebraically determine this function’s \( x \)-intercepts using the quadratic formula. Express your answers in simplest radical form.

(b) Express the \( x \)-intercepts of the quadratic to the nearest hundredth.

(c) Using your calculator, sketch a graph of the quadratic on the axes given. Use the ZERO command on your calculator to verify your answers from part (b). Label the zeros on the graph.

**Exercise #2:** Which of the following sets represents the \( x \)-intercepts of \( y = 3x^2 - 19x + 6 \)?

\[
\begin{align*}
(1) & \quad \left\{ \frac{1}{2}, \frac{7}{3} \right\} \\
(2) & \quad \left\{ \frac{1}{6} + \frac{\sqrt{17}}{2}, \frac{1}{6} - \frac{\sqrt{17}}{2} \right\} \\
(3) & \quad \left\{ 2 - \sqrt{5}, 2 + \sqrt{5} \right\} \\
(4) & \quad \left\{ \frac{1}{3}, 6 \right\}
\end{align*}
\]
**Exercise #3:** (Revisiting the Crazy Carmel Corn Company) – Recall that the Crazy Carmel Corn company modeled the percent of popcorn kernels that would pop, \( P \), as a function of the oil temperature, \( T \), in degrees Fahrenheit using the equation

\[
P = -\frac{1}{250} T^2 + 2.8T - 394
\]

The company would like to find the range of temperatures that ensures that at least 50% of the kernels will pop. Write an inequality whose result is the temperature range the company would like to find. Solve this inequality with the help of the quadratic formula. Round all temperatures to the nearest tenth of a degree.

**Exercise #4:** Find the intersection points of the linear-quadratic system shown below algebraically. Then, use your calculator to help produce a sketch of the system. Label the intersection points you found on your graph.

\[
y = 4x^2 - 6x + 2 \quad \text{and} \quad y = 6x - 3
\]

**Note:** The fact that the solutions to this system were rational numbers indicates that the quadratic equation in Exercise #4 could have been solved using factoring and the Zero Product Law.
MORE WORK WITH THE QUADRATIC FORMULA
COMMON CORE ALGEBRA II HOMEWORK

**FLUENCY**

1. Which of the following represents the solutions to \(x^2 - 4x + 12 = 6x - 2\)?

   (1) \(x = 4 \pm \sqrt{7}\)  
   (2) \(x = 5 \pm \sqrt{11}\)  
   (3) \(x = 5 \pm 22\) \(\sqrt{}\)  
   (4) \(x = 4 \pm \sqrt{13}\)

2. The smaller root, to the nearest *hundredth*, of \(2x^2 - 3x - 1 = 0\) is

   (1) -0.28  
   (2) -0.50  
   (3) 1.78  
   (4) 3.47

3. The \(x\)-intercepts of \(y = 2x^2 + 7x - 30\) are

   (1) \(x = \frac{-7 \pm \sqrt{191}}{2}\)  
   (2) \(x = -3\) and 5  
   (3) \(x = -6\) and \(\frac{5}{2}\)  
   (4) \(x = -3 \pm \sqrt{131}\)

4. Solve the following equation for all values of \(x\). Express your answers in simplest radical form.

   \[4x^2 - 4x - 5 = 8x + 6\]

5. Solve the following equation for all values of \(x\). Express your answers in simplest radical form.

   \[9x^2 = 6x + 4\]
6. Algebraically solve the system of equations shown below. Note that you can use either factoring or the quadratic formula to find the x-coordinates, but the quadratic formula is probably easier.

\[ y = 6x^2 + 19x - 15 \quad \text{and} \quad y = -12x + 15 \]

APPLICATIONS

7. The Celsius temperature, \( C \), of a chemical reaction increases and then decreases over time according to the formula \( C(t) = -\frac{1}{2}t^2 + 8t + 93 \), where \( t \) represents the time in minutes. Use the Quadratic Formula to help determine the amount of time, to the nearest tenth of a minute, it takes for the reaction to reach 110 degrees Celsius.

REASONING

8. For every quadratic there are two roots (or zeroes or \( x \)-intercepts). They are always given by

\[ x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

Determine a formula, in terms of \( b \) and \( a \) for the sum of these two roots.
UNIT #9

COMPLEX NUMBERS

Lesson #1 – Imaginary Numbers
Lesson #2 – Complex Numbers
Lesson #3 – Solving Quadratic Equations with Complex Solutions
Lesson #4 - The Discriminant of a Quadratic
Recall that in the Real Number System, it is not possible to take the square root of a negative quantity because whenever a real number is squared it is non-negative. This fact has a ramification for finding the x-intercepts of a parabola, as Exercise #1 will illustrate.

**Exercise #1:** On the axes below, a sketch of \( y = x^2 \) is shown. Now, consider the parabola whose equation is given in function notation as \( f(x) = x^2 + 1 \).

(a) How is the graph of \( y = x^2 \) shifted to produce the graph of \( f(x) \)?

(b) Create a quick sketch of \( f(x) \) on the axes below.

(c) What can be said about the x-intercepts of the function \( y = f(x) \)?

(d) Algebraically, show that these intercepts do not exist, in the Real Number System, by solving the incomplete quadratic \( x^2 + 1 = 0 \).

Since we cannot solve this equation using Real Numbers, we introduce a new number, called \( i \), the basis of **imaginary numbers**. Its definition allows us to now have a result when finding the square root of a negative real number. Its definition is given below.

**The Definition of the Imaginary Number \( i \)**

\[ i = \sqrt{-1} \]

**Exercise #2:** Simplify each of the following square roots in terms of \( i \).

(a) \( \sqrt{-9} \)  
(b) \( \sqrt{-100} \)  
(c) \( \sqrt{-32} \)  
(d) \( \sqrt{-18} \)
Exercise #2: Solve each of the following incomplete quadratics. Place your answers in simplest radical form.

(a) \(5x^2 + 8 = -12\)  
(b) \(\frac{1}{2}x^2 + 20 = 2\)  
(c) \(2x^2 - 10 = -36\)

Exercise #3: Which of the following is equivalent to \(5i \cdot 6i\) ?

(1) \(30i\)  
(2) \(11i\)  
(3) \(-30\)  
(4) \(-11\)

Powers of \(i\) display a remarkable pattern that allow us to simplify large powers of \(i\) into one of 4 cases. This pattern is discovered in Exercise #4.

Exercise #4: Simplify each of the following powers of \(i\).

\[i^1 = i\quad i^2 = \quad i^3 = \quad i^4 = \]
\[i^5 = \quad i^6 = \quad i^7 = \quad i^8 = \]

We see, then, from this pattern that every power of \(i\) is either \(-1, 1, i,\) or \(-i\). And the pattern will repeat.

Exercise #5: From the pattern of Exercise #4, simplify each of the following powers of \(i\).

(a) \(i^{38} = \)  
(b) \(i^{21} = \)  
(c) \(i^{83} = \)  
(d) \(i^{40} = \)

Exercise #6: Which of the following is equivalent to \(5i^{16} + 3i^{23} + i^{26}\) ?

(1) \(8 + 2i\)  
(2) \(4 - 3i\)  
(3) \(5 - 4i\)  
(4) \(2 + 7i\)
IMAGINARY NUMBERS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. The imaginary number \( i \) is defined as
   - (1) \(-1\)
   - (2) \(\sqrt{-1}\)
   - (3) \(\sqrt{-4}\)
   - (4) \((-1)^2\)

2. Which of the following is equivalent to \(\sqrt{-128}\)?
   - (1) \(8\sqrt{2}\)
   - (2) \(8i\)
   - (3) \(-8\sqrt{2}\)
   - (4) \(8i\sqrt{2}\)

3. The sum \(\sqrt{-9} + \sqrt{-16}\) is equal to
   - (1) 5
   - (2) \(5i\)
   - (3) \(7i\)
   - (4) 7

4. Which of the following powers of \( i \) is not equal to one?
   - (1) \(i^{16}\)
   - (2) \(i^{26}\)
   - (3) \(i^{32}\)
   - (4) \(i^{48}\)

5. Which of the following represents all solutions to the equation \(\frac{1}{3}x^2 + 10 = 7\)?
   - (1) \(x = \pm 3i\)
   - (2) \(x = \pm 5i\)
   - (3) \(x = \pm i\)
   - (4) \(x = \pm 2i\)

6. Solve each of the following incomplete quadratics. Express your answers in simplest radical form.
   - (a) \(2x^2 + 100 = -62\)
   - (b) \(\frac{2}{3}x^2 + 20 = 2\)
7. Which of the following represents the solution set of \( \frac{1}{2}x^2 - 12 = -37 \)?

(1) \( \pm 7i \)  
(2) \( \pm 7i\sqrt{2} \)  
(3) \( \pm 5i\sqrt{2} \)  
(4) \( \pm 3i\sqrt{2} \)

8. Simplify each of the following powers of \( i \) into either \(-1, 1, i, \) or \(-i\).

(a) \( i^2 \)  
(b) \( i^3 \)  
(c) \( i^4 \)  
(d) \( i^{11} \)

(e) \( i^{41} \)  
(f) \( i^{30} \)  
(g) \( i^{25} \)  
(h) \( i^{36} \)

(i) \( i^{51} \)  
(j) \( i^{45} \)  
(k) \( i^{80} \)  
(l) \( i^{70} \)

9. Which of the following is equivalent to \( i^7 + i^8 + i^9 + i^{10} \)?

(1) \( 1 \)  
(2) \( 2 + i \)  
(3) \( 1 - i \)  
(4) \( 0 \)

10. When simplified the sum \( 5i^{18} + 7i^{25} + 2i^{28} + 6i^{43} \) is equal to

(1) \( 2 - 4i \)  
(2) \( -3 + i \)  
(3) \( 5 - 7i \)  
(4) \( 8 + i \)

11. The product \( (6 + 2i)(4 - 3i) \) can be written as

(1) \( 24 - 6i \)  
(2) \( 18 + 10i \)  
(3) \( 2 + 5i \)  
(4) \( 30 - 10i \)
All numbers fall into a very broad category known as complex numbers. Complex numbers can always be thought of as a combination of a real number with an imaginary number and will have the form:

\[ a + bi \]

where \( a \) and \( b \) are real numbers.

We say that \( a \) is the real part of the number and \( bi \) is the imaginary part of the number. These two parts, the real and imaginary, cannot be combined. Like real numbers, complex numbers may be added and subtracted. The key to these operations is that real components can combine with real components and imaginary with imaginary.

**Exercise #1:** Find each of the following sums and differences.

(a) \((-2 + 7i) + (6 + 2i)\)  
(b) \((8i) + (12 - i)\)  
(c) \((5 + 3i) - (2 - 7i)\)  
(d) \((-3 + 5i) - (-8 + 2i)\)

**Exercise #2:** Which of the following represents the sum of \((6 + 2i)\) and \((-8 - 5i)\)?

1. \(5i\)  
2. \(-2 - 3i\)  
3. \(2 + 3i\)  
4. \(-5i\)

Adding and subtracting complex numbers is straightforward because the process is similar to combining algebraic expressions that have like terms. The complex numbers are closed under addition and subtraction, i.e. when you add or subtract two complex numbers the results is a complex number as well. But, is multiplication closed?

**Exercise #3** Find the following products. Write each of your answers as a complex number in the form \( a + bi \).

(a) \((3 + 5i)(7 + 2i)\)  
(b) \((-2 + 6i)(3 - 2i)\)  
(c) \((4 + i)(-5 - 3i)\)
**Exercise #4:** Consider the more general product \((a + bi)(c + di)\) where constants \(a, b, c\) and \(d\) are real numbers.

(a) Show that the real component of this product will always be \(ac - bd\).

(b) Show that the product of \(2 + 3i\) and \(4 - 6i\) results in a purely real number.

(c) Under what conditions will the product of two complex numbers always be a purely imaginary number? Check by generating a pair of complex numbers that have this type of product.

---

**Exercise #5:** Determine the result of the calculation below in simplest \(a + bi\) form.

\[(5 + 2i)(-3 + i) + 4i(2 + 3i)\]

---

**Exercise #6:** Which of the following products would be a purely real number?

(1) \((4 + 2i)(3 - i)\)

(2) \((-3 + i)(-2 + 4i)\)

(3) \((5 + 2i)(5 - 2i)\)

(4) \((6 + 3i)(6 + 3i)\)
COMPLEX NUMBERS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Find each of the following sum or difference.
   (a) $(6+3i) + (-2+9i)$  
   (b) $(-7+i) - (3+5i)$  
   (c) $(10-3i) + (6-8i)$

   (d) $(-2+7i) - (15-6i)$  
   (e) $(15+2i) + (5-5i)$  
   (f) $(-1+i) - (-5-6i)$

2. Which of the following is equivalent to $3(5+2i) - 2(3-6i)$?
   (1) $9+18i$  
   (2) $21+8i$  
   (3) $9-6i$  
   (4) $21-2i$

3. Find each of the following products in simplest $a+bi$ form.
   (a) $(5-2i)(-1+7i)$  
   (b) $(3+9i)(2+4i)$  
   (c) $(-4-i)(-2+6i)$

4. Complex conjugates are two complex numbers that have the form $a+bi$ and $a-bi$. Find the following products of complex conjugates:
   (a) $(5-7i)(5+7i)$  
   (b) $(10+i)(10-i)$  
   (c) $(-3+8i)(-3-8i)$

   (d) What's true about the product of two complex conjugates?
5. Show that the product of \(a + bi\) and \(a - bi\) is the purely real number \(a^2 + b^2\).

6. The product of \((-8 + 2i)\) and its conjugate is equal to

   (1) 64 + 4i  
   (2) 60  
   (3) 68  
   (4) 60 - 4i

7. The complex computation \((6 + 2i)(6 - 2i) - (3 - 4i)(3 + 4i)\) can be simplified to

   (1) 15  
   (2) 39  
   (3) -10  
   (4) -35

8. Perform the following complex calculation. Express your answer in simplest \(a + bi\) form.

   \[(8 + 5i)(3 + 2i) - (4 + i)(4 - i)\]

9. Perform the following complex calculation. Express your answer in simplest \(a + bi\) form.

   \[7(3 - 5i) + (4 - 2i)(-6 + 7i)\]

10. Simplify the following complex expression. Write your answer in simplest \(a + bi\) form.

    \[(5 + 2i)^2 + (2 - i)^2\]
SOLVING QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS
COMMON CORE ALGEBRA II

As we saw in the last unit, the roots or zeroes of any quadratic equation can be found using the quadratic formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Since this formula contains a square root, it is fair to investigate solutions to quadratic equations now when the quantity \( b^2 - 4ac \), known as the **discriminant**, is negative. Up to this point, we would have concluded that if the discriminant was negative, the quadratic had no (real) solutions. But, now it can have **complex solutions**.

**Exercise #1:** Use the quadratic formula to find all solutions to the following equation. Express your answers in simplest \( a + bi \) form.

\[ x^2 - 4x + 29 = 0 \]

As long as our solutions can include complex numbers, then any quadratic equation can be solved for two roots.

**Exercise #2:** Solve each of the following quadratic equations. Express your answers in simplest \( a + bi \) form.

(a) \( x^2 - 5x + 30 = 7x - 10 \)  
(b) \( x^2 + 16x + 15 = 10x + 4 \)
There is an interesting connection between the \( x \)-intercepts (zeroes) of a parabola and complex roots with non-zero imaginary parts. The next exercise illustrates this important concept.

**Exercise #3:** Consider the parabola whose equation is \( y = x^2 - 6x + 13 \).

(a) Algebraically find the \( x \)-intercepts of this parabola. Express your answers in simplest \( a + bi \) form.

(b) Using your calculator, sketch a graph of the parabola on the axes below. Use the window indicated.

(c) From your answers to (a) and (b), what can be said about parabolas whose zeroes are complex roots with non-zero imaginary parts?

**Exercise #4:** Use the discriminant of each of the following quadratics to determine whether it has \( x \)-intercepts.

(a) \( y = x^2 - 3x - 10 \)  
(b) \( y = x^2 + 6x + 10 \)  
(c) \( y = 2x^2 + 3x + 5 \)

**Exercise #5:** Which of the following quadratic functions, when graphed, would not cross the \( x \)-axis?

(1) \( y = 2x^2 + 5x - 3 \)  
(2) \( y = -x^2 - x + 6 \)  
(3) \( y = 4x^2 - 4x + 5 \)  
(4) \( y = 3x^2 - 13x + 4 \)
SOLVING QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following quadratic equations. Express your solutions in simplest \( a + bi \) form. Check.

   (a) \( x^2 + 4x + 20 = 12x - 5 \)
   (b) \( x^2 = x - 1 \)

   (c) \( 2x^2 - 25x + 27 = -15x - 10 \)
   (d) \( 8x^2 + 36x + 24 = 12x + 5 \)

   (e) \( x^2 + 6x + 15 = 8x - 2 \)
   (f) \( 4x^2 + 38x + 50 = 10x - 35 \)
2. Which of the following represents the solution set to the equation \( x^2 - 2x + 2 = 0 \)?

(1) \( x = -1 \) or \( 2 \)  
(2) \( x = 1 \pm 2i \) 
(3) \( x = 2 \pm i \)  
(4) \( x = 1 \pm i \)

3. The solutions to the equation \( x^2 + 6x + 11 = 0 \) are

(1) \( x = -3 \pm i\sqrt{2} \)  
(2) \( x = -3 \pm 2i\sqrt{2} \)  
(3) \( x = -6 \pm i\sqrt{11} \)  
(4) \( x = -6 \pm 2i\sqrt{11} \)

4. Using the determinant, \( b^2 - 4ac \), determine whether each of the following quadratics has real or imaginary zeroes.

(a) \( y = 2x^2 - 7x + 6 \)  
(b) \( y = 3x^2 + 2x + 1 \)  
(c) \( y = x^2 - 8x + 14 \)

(d) \( y = 2x^2 - 12x + 26 \)  
(e) \( y = -2x^2 + 6x - 5 \)  
(f) \( y = 4x^2 - 4x + 1 \)

5. Which of the following quadratics, if graphed, would lie entirely above the \( x \)-axis? Try to use the discriminant to solve this problem and then graph to check.

(1) \( y = 2x^2 + x - 21 \)  
(2) \( y = x^2 - x - 6 \)  
(3) \( y = x^2 - 4x + 7 \)  
(4) \( y = x^2 - 10x + 16 \)

6. For what values of \( c \) will the quadratic \( y = x^2 + 6x + c \) have no real zeroes? Set up and solve an inequality for this problem.

**REASONING**
THE DISCRIMINANT OF A QUADRATIC
COMMON CORE ALGEBRA II

Since the roots of a quadratic can be found using \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), if \( a, b, \) and \( c \) are all rational numbers, the quantity under the square root, \( b^2 - 4ac \), truly dictates what type of numbers the roots of a quadratic (and its \( x \)-intercepts) turn out to be. It reduces down to four cases which will be explored in Exercise #1.

Exercise #1: For each of the following quadratics, calculate its discriminant, its roots, and state the number and nature (whether they are rational, irrational or imaginary) of the roots.

(a) Case I – The Discriminant is a Perfect Square \(- x^2 + 3x - 10 = 0 \).
\[ D = b^2 - 4ac = \]

Roots:

Number and Nature:

(b) Case II – The Discriminant is Not a Perfect Square \(- x^2 - 6x + 7 = 0 \).
\[ D = b^2 - 4ac = \]

Roots:

Number and Nature:

(c) Case III – The Discriminant is Equal to Zero \(- x^2 - 10x + 25 = 0 \).
\[ D = b^2 - 4ac = \]

Roots:

Number and Nature:

(d) Case IV – The Discriminant is Less than Zero \(- x^2 - 8x + 20 = 0 \).
\[ D = b^2 - 4ac = \]

Roots:

Number and Nature:
In the last lesson, we explored Case IV extensively. In the case where the discriminant is negative, the roots of the quadratic are imaginary and it does not have x-intercepts (i.e. it does not cross the x-axis).

**Exercise #2:** By using only the discriminant, determine the number and nature of the roots of each of the following quadratics.

(a) \(2x^2 + 7x - 4 = 0\)  
(b) \(x^2 - 8x + 25 = 0\)  
(c) \(4x^2 + 4x + 1 = 0\)

(d) \(x^2 + 6x + 15 = 0\)  
(e) \(4x^2 - 4x - 7 = 0\)  
(f) \(3x^2 - 7x + 2 = 0\)

**Exercise #3:** Consider the quadratic function whose equation is \(y = x^2 - 4x + 4\). Determine the number of x-intercepts of this quadratic from the value of its discriminant. Then, sketch its graph on the axes given. We say that this parabola is tangent to the x-axis.

**Exercise #4:** Which of the following parabolas has two unequal, rational x-intercepts?

1. \(y = x^2 - 2x - 1\)
2. \(y = x^2 + 2x - 15\)
3. \(y = x^2 - 8x + 16\)
4. \(y = x^2 - 3x + 5\)

**Exercise #5:** For what values of \(a\) will the parabola \(y = ax^2 + 4x + 2\) not cross the x-axis?
THE DISCRIMINANT OF A QUADRATIC
COMMON CORE ALGEBRA II HOMEWORK

SKILLS

1. For each of the following quadratic equations, determine the number and the nature of the roots by first calculating the quadratic’s discriminant.

(a) \(2x^2 + 4x + 5 = 0\)
(b) \(9x^2 - 12x + 4 = 0\)
(c) \(4x^2 - 13x + 3 = 0\)

(d) \(x^2 + 8x + 11 = 0\)
(e) \(4x^2 + 4x - 7 = 0\)
(f) \(36x^2 - 12x + 1 = 0\)

(g) \(-3x^2 + 4x - 8 = 0\)
(h) \(3x^2 + 8x + 4 = 0\)
(i) \(x^2 + 8x + 41 = 0\)

2. The roots of \(2x^2 - 4x - 7 = 0\) are

(1) unequal and rational  (3) unequal and irrational
(2) unequal and imaginary  (4) equal and rational

3. Which of the following quadratics has imaginary roots?

(1) \(x^2 + 3x - 5 = 0\)  (3) \(2x^2 - 3x + 1 = 0\)
(2) \(x^2 + 6x + 10 = 0\)  (4) \(x^2 - 7x + 10 = 0\)

4. Which of the following quadratic, when graphed, would touch the \(x\)-axis exactly once?

(1) \(y = x^2 - 2x - 3\)  (3) \(y = x^2 + 5x - 2\)
(2) \(y = 2x^2 + 3x + 7\)  (4) \(y = x^2 - 12x + 36\)
5. If graphed, which of the following parabolas would lie entirely below the $x$-axis?

(1) $y = x^2 + 5x + 10$  
(2) $y = -2x^2 - 5x + 3$  
(3) $y = -2x^2 + 6x - 5$  
(4) $y = x^2 + 6x + 9$

6. Which parabola below, when graphed, would cross the $x$-axis at two unequal irrational locations?

(1) $y = 2x^2 + 11x + 12$  
(2) $y = x^2 + 2x - 4$  
(3) $y = 9x^2 - 6x + 1$  
(4) $y = 2x^2 + 4x + 9$

**REASONING**

7. Determine all values of $a$ that will cause the parabola $y = ax^2 + 10x + 1$ to cross the $x$-axis at two distinct locations.

8. Consider the parabola whose equation is $y = x^2 - 4x$ and the line whose equation is $y = 2x + b$, where $b$ is some unknown constant. Determine the value of $b$ such that the line and the parabola will intersect at exactly one location. Then, sketch the system of equations on the axes below. Label their intersection point.
UNIT #10

POLYNOMIAL AND RATIONAL FUNCTIONS

Lesson #1 – Power Functions
Lesson #2 – Graphs and Zeroes of a Polynomial
Lesson #3 – Creating Polynomial Equations
Lesson #4 – Polynomial Identities
Lesson #5 – Introduction to Rational Functions
Lesson #6 – Simplifying Rational Expressions
Lesson #7 – Multiplying and Dividing Rational Expressions
Lesson #8 – Combining Rational Expressions Using Addition and Subtraction
Lesson #9 – Complex Fractions
Lesson #10 – Polynomial Long Division
Lesson #11 – The Remainder Theorem
Lesson #12 – Solving Rational Equations
Lesson #13 – Solving Rational Inequalities
Lesson #14 – Reasoning About Radical and Rational Equations
Before we start to analyze polynomials of degree higher than two (quadratics), we first will look at very simple functions known as power functions. The formal definition of a power function is given below:

**Power Functions**

Any function of the form: \( f(x) = ax^b \) where \( a \) and \( b \) are real numbers not equal to zero.

**Exercise #1:** For each of the following power functions, state the value of \( a \) and \( b \) by writing the equation in the form \( y = ax^b \).

(a) \( y = \frac{3}{x^2} \)  
(b) \( y = \frac{1}{7x^3} \)  
(c) \( y = 8\sqrt{x} \)  
(d) \( y = \frac{6}{\sqrt[3]{x}} \)

The characteristics of power functions depend on both the value of \( a \) and the value of \( b \). The most important, though, is the exponent (the \( a \) is simply a vertical stretch of the power function).

**Exercise #2:** Consider the general power function \( y = ax^b \).

(a) What can be said about the \( y \)-intercept of any power function if \( b > 0 \)? Illustrate.  
(b) What can be said about the \( y \)-intercept of any power function if \( b < 0 \)? Illustrate.

For now we will just concentrate on power function where the exponent is a positive whole number.

**Exercise #3:** Using your table, fill in the following values for common power functions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
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</tr>
<tr>
<td>( x^3 )</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^4 )</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^5 )</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
From the previous exercise, we should note that when the power function has an even exponent, then positive and negative INPUTS have the same value. When the power function has an odd exponent, then positive and negative inputs have opposite outputs. Recall this is the definition of **even and odd functions**.

**Exercise #4:** Using your calculators, sketch the power functions below using the standard viewing window.

(a) \( y = x^2 \)  
(b) \( y = x^3 \)  
(c) \( y = x^4 \)  
(d) \( y = x^5 \)

**Exercise #5:** Which of the following power functions is shown in the graph below? Explain your choice. Do without the use of your calculator.

1. \( y = -4x^7 \)
2. \( y = -3x^{10} \)
3. \( y = 6x^8 \)
4. \( y = 5x^9 \)

The End Behavior of Polynomials – The behavior of polynomials as the input variable gets very large, both positive and negative, is important to understand. We will explore this in the next exercise.

**Exercise #6:** Consider the two functions \( y_1 = x^3 - 2x^2 - 29x + 30 \) and \( y_2 = x^3 \).

(a) Graph these functions using \( x_{\text{min}} = -10, x_{\text{max}} = 10, y_{\text{min}} = -100, y_{\text{max}} = 100 \)

(b) Graph these functions using \( x_{\text{min}} = -20, x_{\text{max}} = 20, y_{\text{min}} = -1000, y_{\text{max}} = 1000 \)

(c) Graph these functions using \( x_{\text{min}} = -50, x_{\text{max}} = 50, y_{\text{min}} = -10000, y_{\text{max}} = 10000 \)

(d) Graph these functions using \( x_{\text{min}} = -100, x_{\text{max}} = 100, y_{\text{min}} = -100000, y_{\text{max}} = 100000 \)

(e) What do you observe about the nature of the two graphs as the viewing window gets larger?

(f) Why is this occurring?

The end behavior (also known as long-run) of any polynomial is dictated by its highest powered term!!!
POWER FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. **Without** using your calculator, determine which of the following equations could represent the graph shown below. Explain your choice.

   (1) \( y = x^2 \)  
   (2) \( y = x^3 \)  
   (3) \( y = -x^4 \)  
   (4) \( y = -x^5 \)

2. Identify which of the following are power functions. For each that is a power function, write it in the form \( y = ax^n \), where \( a \) and \( n \) are real numbers. Placing them in these forms may take some mindful algebraic manipulation.

   (a) \( y = 3\sqrt{x} \)  
   (b) \( y = 4x^5 - 7 \)  
   (c) \( y = \frac{10}{x^5} \)  
   (d) \( y = \frac{6x^7}{2x^3} \)

   (e) \( y = x^2 + 2x - 7 \)  
   (f) \( y = \sqrt{48x^7} \)  
   (g) \( y = \sqrt[5]{25}x^4 \)  
   (f) \( y = 2(x - 3)^2 \)

3. If the point \((-3, 8)\) lies on the graph of a power function with an even exponent, which of the following points must also lie its graph?

   (1) \((-3, -8)\)  
   (3) \((-3, -8)\)  
   (2) \((3, 8)\)  
   (4) \((8, -3)\)
4. If the point (-5, 12) lies on the graph of a power function with an odd exponent, which of the following points must also lie on its graph?

(1) (5, -12)  (2) (12, -5)

(3) (-5, 12)  (4) (-12, 5)

5. For each of the following polynomials, give a power function that best represents the end behavior of the polynomial.

(a) \( y = 3x^3 - 2x + 12 \)  
(b) \( y = 10 - 8x^2 \)  
(c) \( y = 6x^5 - 4x^3 + x - 120 \)

(d) \( y = -3x^5 + 2x^4 - 4x + 7 \)  
(e) \( y = 5x^4 + 2x^2 \)  
(f) \( y = -4x^5 + 8x^7 - 2x^3 + 3 \)

6. The graph below could be the long-run behavior for which of the following functions? Do this problem without graphing each of the following equations.

(1) \( y = 2x^2 - 7x + 1 \)

(2) \( y = 4x^3 + 2x^2 - 6x + 4 \)

(3) \( y = -5x^4 + 3x^3 - 2x^2 + x + 9 \)

(4) \( y = -3x^5 - 4x^2 + 2x + 1 \)

REASONING

7. Let's examine why end-behavior works a little more closely. Consider the functions \( f(x) = x^3 \) and \( g(x) = x^3 + 2x^2 + 7x + 10 \).

(a) Fill out the table below for the values of \( x \) listed. Round your final column to the nearest hundredth.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( \frac{f(x)}{g(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>10</td>
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<td>50</td>
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<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
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</tr>
</tbody>
</table>

(b) What number is the ratio in the fourth column approaching as \( x \) gets larger? What does this tell you about the part of \( g(x) \) that can be attributed to the cubic term?
A polynomial is a function consisting of terms that all have whole number powers. In its most general form, a polynomial can be written as:

\[ y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

Quadratic and linear functions are the simplest of all polynomials. In this lesson we will explore cubic and quartic functions, those whose highest powers are \( x^3 \) and \( x^4 \) respectively.

**Exercise #1:** For each of the following cubic functions, sketch the graph and circle its \( x \)-intercepts.

(a) \( y = x^3 - 3x^2 - 6x + 8 \)

(b) \( y = 2x^3 - 8x + 9 \)

(c) \( y = 2x^3 - 12x^2 + 18x \)

Clearly, a cubic may have one, two or three real roots and can have two turning points. Just as with parabolas, there exists a tie between a cubic’s factors and its \( x \)-intercepts.

**Exercise #2:** Consider the cubic whose equation is \( y = x^3 - x^2 - 12x \).

(a) *Algebraically* determine the zeroes of this function.

(b) Sketch a graph of this function on the axes below illustrating your answer to part (a).
**Exercise #3:** The largest root of \( x^3 - 9x^2 + 12x + 22 = 0 \) falls between what two consecutive integers?

1. 4 and 5  
2. 6 and 7  
3. 10 and 11  
4. 8 and 9

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**Exercise #4:** Consider the quartic function \( y = x^4 - 5x^2 + 4 \).

(a) Algebraically determine the \( x \)-intercepts of this function.

(b) Verify your answer to part (a) by sketching a graph of the function on the axes below.

---

**Exercise #5:** Consider the quartic whose equation is \( y = x^4 + 3x^3 - 35x^2 - 39x + 70 \).

(a) Sketch a graph of this quartic on the axes below. Label its \( x \)-intercepts.

(b) Based on your graph from part (a), write the expression \( x^4 + 3x^3 - 35x^2 - 39x + 70 \) in its factored form.
Graphs and Zeros of a Polynomial

Common Core Algebra II Homework

Fluency

1. Consider the cubic function \( y = x^3 + 2x^2 - 8x \).
   
   (a) Algebraically determine the zeroes of this cubic function.
   
   (b) Sketch the function on the axes given. Clearly plot and label each \( x \)-intercept.

2. Consider the cubic function \( y = x^3 + 2x^2 - 36x - 72 \).
   
   (a) Find an appropriate \( y \)-window for the \( x \)-window shown on the axes and sketch the graph. Make sure the window is sufficiently large to show the two turning points and all intercepts. Clearly label all \( x \)-intercepts.
   
   (b) What are the solutions to the equation \( x^3 + 2x^2 - 36x - 72 = 0 \)?

   (c) Based on your answers to (b), how must the expression \( x^3 + 2x^2 - 36x - 72 \) factor?
3. Consider the cubic function given by \( y = x^3 - 6x^2 + 12x - 5 \).

   (a) Sketch a graph of this function on the axes given below. Clearly mark all \( x \)-intercepts.

   (b) Considering the graphs of cubics you saw in class and those in problems 1 and 2, what is different about the way this cubic’s graph looks compared to the others?

   ![Graph of y = x^3 - 6x^2 + 12x - 5](image)

4. Consider the quartic function \( y = x^4 - x^3 - 27x^2 + 25x + 50 \).

   (a) Sketch the graph of this function on the axes given below. Clearly mark all \( x \)-intercepts.

   (b) Use your graph from part (a) to solve the equation \( x^4 - x^3 - 27x^2 + 25x + 50 = 0 \).

   (c) Considering your answer to (b), how does the expression \( x^4 - x^3 - 27x^2 + 25x + 50 \) factor?

   ![Graph of y = x^4 - x^3 - 27x^2 + 25x + 50](image)

5. In general, how does the number of zeroes (or \( x \)-intercepts) relate to the highest power of a polynomial? Be specific. Can you make a statement about the minimum number of zeroes as well as the maximum?
The connection between the zeroes of a polynomial and its factors should now be clear. This connection can be used to create equations of polynomials. The key is utilizing the factored form of a polynomial.

**The Factored Form of a Polynomial**

If the set $\{r_1, r_2, r_3, \ldots, r_n\}$ represent the roots (zeroes) of a polynomial, then the polynomial can be written as:

$$y = a(x - r_1)(x - r_2)\cdots(x - r_n)$$

where $a$ is some constant determined by another point.

**Exercise #1:** Determine the equation of a quadratic function whose roots are $-3$ and $4$ and which passes through the point $(2, -50)$. Express your answer in standard form ($y = ax^2 + bx + c$). Verify your answer by creating a sketch of the function on the axes below.

![Graph](image)

It’s important to understand how the $a$ value affects the graph of the polynomial. This is easiest to explore if the polynomial remains in factored form.

**Exercise #2:** Consider quadratic polynomials of the form $y = a(x + 2)(x - 5)$, where $a \neq 0$.

(a) What are the $x$-intercepts of this parabola?

(b) Sketch on the axes given the following equations:

$$y = (x + 2)(x - 5)$$
$$y = 2(x + 2)(x - 5)$$
$$y = 4(x + 2)(x - 5)$$

![Graph](image)
As we can see from this exercise, the value of $a$ does not change the zeroes of the function, but does vertically stretch the function. We can create equations of higher powered polynomials in a similar fashion.

**Exercise #3:** Create the equation of the cubic, in standard form, that has $x$-intercepts of $-4, 2,$ and $5$ and passes through the point $(6, 20)$. Verify your answer by sketching the cubic’s graph on the axes below.

**Exercise #4:** Create the equation of a cubic in standard form that has a double zero at $-2$ and another zero at $4$. The cubic has a $y$-intercept of $16$. Sketch your cubic on the axes below to verify your result.

**Exercise #5:** How would you describe this cubic curve at its double root?
CREATING EQUATIONS OF POLYNOMIALS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Create the equation of a quadratic polynomial, in standard form, that has zeroes of \(-5\) and \(2\) and which passes through the point \((3, -24)\). Sketch the graph of the quadratic below to verify your result.

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3. Create an equation for a cubic function, in standard form, that has $x$-intercepts given by the set $\{-3, 1, 7\}$ and which passes through the point $(-2, 54)$. Sketch your result on the axes shown below.

4. Create the equation of a cubic whose $x$-intercepts are given by the set $\{-6, -3, 5\}$ and which passes through the point $(3, 36)$. Note that your leading coefficient in this case will be a non-integer. Sketch your result below.
Polynomials are expressions consisting of addition and subtraction of variables and coefficients all raised to non-negative, integer powers. As in the last few lessons, there can be a single variable or multiple variables.

**Exercise #1:** Which of the following is not a polynomial expression? Explain why your choice fails to be a polynomial.

1. \(x^3 + 2x^2y + y^3\)
2. \(x^7 + y^7\)
3. \(x^4y^2 + 2xy^{1/2}\)
4. \(x^3 - y^2\)

Because polynomials consist of basic operations on variables, they can be manipulated using the associative, commutative, and distributive properties (as you have done many times). These operations can result in what are known as **polynomial identities**. An identity is defined more broadly below:

### IDENTITIES

An identity is an equation that is true for all values of the replacement variable or variables.

**Exercise #2:** One identity that you should be familiar with is \(x^2 - y^2 = (x-y)(x+y)\).

(a) Test this identity with the pair \(x=10\) and \(y=3\).

(b) Prove this identity by manipulating the right side of the equation.

(c) Use this identity to evaluate the difference \(50^2 - 49^2\).

(d) Use this identify to simplify and then evaluate the product \((51)(49)\).

**Exercise #3:** Prove the identity \((a+b)^2 = a^2 + 2ab + b^2\) and use it to evaluate \(35^2\).
Sometimes identities can have geometric connections as well as algebraic. The Pythagorean Theorem gives us ample identities.

**Exercise #4:** A right triangle is shown below whose sides are $2x$, $x^2 - 1$, and $x^2 + 1$.

(a) Show that these will be the side lengths of a right triangle as long as $x > 1$, i.e. show that

\[(2x)^2 + (x^2 - 1)^2 = (x^2 + 1)^2\]

is an identity.

(b) Based on our work from (a) and on the triangle shown, explain why any even integer (other than 2) must be part of a Pythagorean triple, i.e. a set of 3 integers that could be the sides of a right triangle. Generate a Pythagorean triple that has 10 in it and a separate one that has 14 in it.

**Exercise #5:** Consider the polynomial identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

(a) Prove this identity by expanding the left-hand side of the equation.

(b) Use your calculator to find the value of $11^3$ then use the identity to show the same result. Carefully consider your choice of $a$ and $b$. 
1. One of the two expressions below is an identity and one of them is not. Determine which is an identity by testing the truth value of the equation for various values of $x$. Show the values of $x$ that you test. Remember, an identity will be true for every value of $x$.

Equation #1: $(x + 1)^2 = 4x + 4$
Equation #2: $(x + 2)^2 = x^2 + 4x + 4$

2. Which of the following equations represents an identity?

(1) $2x + 1 = 3x + 4$
(2) $6x + 3 = 2x + 10$
(3) $4x - 3 = 2(2x + 7)$
(4) $4(5x + 2) = 20x + 8$

3. One of the more useful identities that students almost inherently learn is:

$$(x + c)(x + d) = x^2 + (c + d)x + cd$$

(a) Prove this identity. You may choose to algebraically manipulate one or both sides of the equation to justify the equivalence.

(b) This identity allows you to multiply common binomials very quickly. Find the following products in simplest trinomial form.

(i) $(x + 3)(x + 7)$
(ii) $(x - 7)(x - 2)$
(iii) $(x + 10)(x - 3)$
4. You should be well aware of the difference of perfect squares, i.e. \( x^2 - y^2 = (x - y)(x + y) \). But there is also an identity for the difference of perfect cubes:

\[
x^3 - y^3 = (x - y)(x^2 + xy + y^2)
\]

(a) Prove this identity by expanding the product on the right-hand side of the equation.

(b) Use the identity to find the value of \( 10^3 - 9^3 \) without the use of your calculator. Show the steps in your calculation. Then, verify with your calculator.

APPLICATIONS

5. Another famous identity that can be used to generate Pythagorean Triples is shown below:

\[
(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2
\]

The complicated sides are shown on the diagram.

(a) Prove this identity by expanding the products on both sides of the equation.

(b) Generate the sides of the right triangle if \( x = 4 \) and \( y = 1 \). Show that these sides satisfy the Pythagorean Theorem.

(c) **Reasoning:** This relationship leads to the conclusion that there is no Pythagorean triple of this form that contains the integer 2. Why?
**INTRODUCTION TO RATIONAL FUNCTIONS**

**COMMON CORE ALGEBRA II**

Rational functions are simply the ratio of polynomial functions. They take on more interesting properties and have more interesting graphs than polynomials because of the interaction between the numerator and denominator of the fraction. In Common Core Algebra II, we will be primarily concerned with the algebra of these functions. But in this lesson we will explore some of their characteristics.

**Exercise #1:** Consider the rational function given by \( f(x) = \frac{x+6}{x-3} \).

(a) Algebraically determine the \( y \)-intercept for this function.

(b) Algebraically determine the \( x \)-intercept of this function. Hint – a fraction can only equal zero if its numerator is zero.

(c) For what value of \( x \) is this function undefined? Why is it undefined at this value?

(d) Based on (c), state the domain of this function in set-builder notation.

**Exercise #2:** Find all values of \( x \) for which the rational function \( h(x) = \frac{x+5}{2x^2 + 11x - 6} \) is undefined. Verify by using your calculator to evaluate this expression for these values.

**Exercise #3:** Which of the following represents the domain of the function \( f(x) = \frac{x-3}{x^2 - 6x - 16} \)?

- (1) \( \{ x \mid x \neq \pm 4 \} \)
- (2) \( \{ x \mid x \neq 3 \} \)
- (3) \( \{ x \mid x \neq -2 \text{ and } 8 \} \)
- (4) \( \{ x \mid x \neq -6 \text{ and } 3 \} \)
**Exercise #4:** If \( g(x) = 3x - 2 \) and \( f(x) = \frac{2x + 1}{x + 5} \) then find:

(a) \( f(g(-2)) \)  
(b) \( f(g(2)) \)  
(c) \( f(g(x)) \)

**Exercise #5:** Find formulas for the inverse of each of the following simple rational functions below. Recall that as a first step, switch the roles of \( x \) and \( y \).

(a) \( y = \frac{x}{x - 2} \)  
(b) \( y = \frac{x + 3}{2x} \)

**Exercise #6:** The function \( f(x) = \frac{x^2 - 8}{4x} \) is either an even or an odd function. Determine which it is and justify.

Based on your answer, what type of symmetry must this function have? Use your calculator to sketch a graph to verify.
INTRODUCTION TO RATIONAL FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following values of \(x\) is \textit{not} in the domain of \(f(x) = \frac{x+3}{x-7}\)?
   \(\text{(1) } x = -7\) \hspace{1cm} \(\text{(3) } x = 3\)
   \(\text{(2) } x = 7\) \hspace{1cm} \(\text{(4) } x = -3\)

2. Which of the following values of \(x\) is \textit{not} in the domain of \(g(x) = \frac{4x-1}{2x+1}\)?
   \(\text{(1) } x = -\frac{1}{2}\) \hspace{1cm} \(\text{(3) } x = \frac{1}{4}\)
   \(\text{(2) } x = -1\) \hspace{1cm} \(\text{(4) } x = -3\)

3. Which values of \(x\), when substituted into the function \(y = \frac{x-4}{2x^2+8x}\), would make it undefined?
   \(\text{(1) } x = 2\) and \(8\) \hspace{1cm} \(\text{(3) } x = -4\) and \(4\)
   \(\text{(2) } x = -4\) and \(8\) \hspace{1cm} \(\text{(4) } x = -4\) and \(0\)

4. Which of the following represents the domain of \(y = \frac{x^2-4}{x^2+5x-14}\)?
   \(\text{(1) } \{x \mid x \neq \pm 2\}\) \hspace{1cm} \(\text{(3) } \{x \mid x \neq -4 \text{ and } 14\}\)
   \(\text{(2) }\{x \mid x \neq -7 \text{ and } 2\}\) \hspace{1cm} \(\text{(4) }\{x \mid x \neq -5 \text{ and } 14\}\)

5. Which of the following represents the domain of \(g(x) = \frac{3x-1}{2x^2-x-10}\)?
   \(\text{(1) } \{x \mid x \neq \frac{1}{3}\}\) \hspace{1cm} \(\text{(3) } \{x \mid x \neq -\frac{1}{2} \text{ and } 5\}\)
   \(\text{(2) }\{x \mid x \neq -\frac{1}{3} \text{ and } \frac{1}{2}\}\) \hspace{1cm} \(\text{(4) }\{x \mid x \neq -2 \text{ and } \frac{5}{2}\}\)

6. If \(f(x) = 2x+7\) and \(g(x) = \frac{x^2-4}{2x+1}\) then \(g(f(-5)) = ?\)
   \(\text{(1) } -1\) \hspace{1cm} \(\text{(3) } 6\)
   \(\text{(2) } 2\) \hspace{1cm} \(\text{(4) } -3\)
7. If \( f(x) = \frac{3x-2}{2x} \) and \( g(x) = 4x-1 \) then \( f\left(g\left(x\right)\right) = ? \)

(1) \( \frac{7x-3}{2x} \)  
(2) \( \frac{12x-9}{8x-2} \)  
(3) \( \frac{12x-5}{8x-2} \)  
(4) \( \frac{5x-4}{x} \)  

8. The \( y \)-intercept of the rational function \( y = \frac{2x+15}{x-3} \) is

(1) 15  
(2) -5  
(3) -3  
(4) 12  

9. Find formulas for the inverse of each of the following rational functions.

(a) \( y = \frac{5x}{x-2} \)  
(b) \( y = \frac{3x+2}{x+4} \)  

10. Consider the rational function \( y = \frac{9-x^2}{x^2+1} \).

(a) Find the function’s \( y \)-intercept algebraically.  
(c) Sketch the function on the axes below. Clearly label your \( x \) and \( y \) intercepts.

(b) Find the function’s \( x \)-intercepts algebraically.

(d) Is this an even or an odd function? Explain graphically.
SIMPLIFYING RATIONAL EXPRESSIONS
COMMON CORE ALGEBRA II

Simplifying a rational expression into its lowest terms is an extremely useful skill. Its algebra is based on how we simply numerical fractions. The basic principle is developed in the first exercise.

**Exercise #1:** Recall that to multiply fractions, one simply multiplies their numerators and denominators.

(a) Simplify the numerical fraction \( \frac{18}{12} \) by first expressing it as a product of two fractions, one of which is equal to one.

(b) Simplify the algebraic fraction \( \frac{x^2 - 9}{2x + 6} \) by first expressing it as the product of two fractions (factor!), one of which is equal to one.

Every time we simplify a fraction, we are essentially finding all common factors of the numerator and denominator and dividing them to be equal to one. Key in this process is that the numerator and denominator **must be factored** and **only common factors cancel each other**. This is true whether our fraction contains monomial, binomial, or polynomial expressions.

**Exercise #2:** Simplify each of the following monomials dividing other monomials.

(a) \( \frac{3x^5 y^6}{6x^8 y^3} \)  
(b) \( \frac{20x^{10} y^8}{4x^2} \)  
(c) \( \frac{7x^3 y}{21x^5 y^8} \)

**Exercise #3:** Which of the following is equivalent to \( \frac{10x^6 y^3}{15x^2 y^6} \)?

1. \( \frac{2x^3}{3y^2} \)  
2. \( \frac{3x^8}{2y^9} \)  
3. \( \frac{2x^4}{3y^3} \)  
4. \( \frac{3x^2}{2y^3} \)
When simplifying rational expressions that are more complex, always factor first, then identify common factors that can be eliminated.

**Exercise #4:** Simplify each of the following rational expressions.

(a) \( \frac{x^2 + 5x - 14}{x^2 - 4} \)

(b) \( \frac{4x^2 - 1}{10x - 5} \)

(c) \( \frac{3x^2 + 14x + 8}{x^2 - 16} \)

A special type of simplifying occurs whenever expressions of the form \((x - y)\) and \((y - x)\) are involved.

**Exercise #5:** Simplify each of the following fractions.

(a) \( \frac{9 - 6}{6 - 9} \)

(b) \( \frac{15 - 3}{3 - 15} \)

(c) \( \frac{a - b}{b - a} \)

**Exercise #6:** Which of the following is equivalent to \( \frac{2x - 10}{25 - x^2} \)?

1. \( \frac{-2}{x + 5} \)
2. \( \frac{2 - x}{5} \)
3. \( \frac{x + 5}{2} \)
4. \( \frac{2}{x - 5} \)

**Exercise #7:** Which of the following is equivalent to \( \frac{x^2 - 6x + 9}{18 - 6x} \)?

1. \( \frac{-x - 3}{6} \)
2. \( \frac{x - 3}{6} \)
3. \( \frac{x + 3}{9} \)
4. \( \frac{3 - x}{6} \)
SIMPLIFYING RATIONAL EXPRESSIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Write each of the following ratios in simplest form.

   (a) \( \frac{5x^8}{20x^2} \)  
   (b) \( \frac{-12y^3}{8y^{12}} \)  
   (c) \( \frac{6x^{10}y^2}{15x^4y^5} \)  
   (d) \( \frac{24x^3y^7}{12x^6y^{10}} \)

2. Which of the following is equivalent to the expression \( \frac{4x^6y^4}{12x^2y^8} \)?

   (1) \( \frac{x^4}{3y^2} \)  
   (2) \( \frac{3y^2}{x^4} \)  
   (3) \( \frac{3x^3}{y^2} \)  
   (4) \( \frac{x^3}{3y^2} \)

3. Simplify each of the following rational expressions.

   (a) \( \frac{x^2 - 25}{4x - 20} \)  
   (b) \( \frac{x^2 + 11x + 24}{x^2 - 9} \)  
   (c) \( \frac{4x^2 - 1}{5x - 10x^2} \)

   (d) \( \frac{9x^2 - 4}{3x^2 + 4x - 4} \)  
   (e) \( \frac{7x^2 - 42x}{x^2 + 2x - 48} \)  
   (f) \( \frac{2x^2 - 3x - 5}{25 - 4x^2} \)
4. Which of the following is equivalent to the fraction \( \frac{x^2 - 9x + 18}{15x - 5x^2} \)?

(1) \( \frac{x-3}{5x} \)  
(2) \( \frac{x+6}{5x} \)  
(3) \( \frac{6-x}{5x} \)  
(4) \( \frac{-x-6}{5x} \)

5. The rational expression \( \frac{2x^2 + 7x + 6}{x^2 - 4} \) can be equivalently rewritten as

(1) \( \frac{2x+3}{x-2} \)  
(2) \( \frac{2x+1}{x-6} \)  
(3) \( \frac{2x-3}{2-x} \)  
(4) \( \frac{3-2x}{x+2} \)

6. Written in simplest form, the fraction \( \frac{y^2 - x^2}{5x-5y} \) is equal to

(1) \( 5y - 5x \)  
(2) \( \frac{y-x}{5} \)  
(3) \( \frac{-\left(x+y\right)}{5} \)  
(4) \( \frac{x-y}{5} \)

**REASONING**

7. When we simplify an algebraic fraction, we are producing equivalent expressions for most values of \( x \).

Consider the expressions \( \frac{x^2 - 4}{2x - 4} \) and \( \frac{x+2}{2} \).

(a) Show by simplifying the first expression that these two are equivalent.

(b) Use your calculator to fill out the value for both of these expressions to show their equivalence.

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(c) Clearly these two expressions are not equivalent for an input value of \( x = 2 \). Explain why.
MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS
COMMON CORE ALGEBRA II

Multiplication of rational expressions follows the same principles as those involved in simplifying them. The process is illustrated in Exercise #1 with both a numerical and algebraic fraction. Notice the parallels.

Exercise #1: Simplify each of the following rational expressions by factoring completely. For the numerical fraction, make sure to prime factor all numerators and denominators.

(a) \( \frac{6}{8} \cdot \frac{10}{3} \)  
(b) \( \frac{x^2 - 4}{x^2 - x - 6} \cdot \frac{3x^2 + 15x}{6x^2 - 12x} \)

The ability to “cross-cancel” with fractions is a result of the two facts: (1) to multiply fractions we multiply their respective numerators and denominators and (2) multiplication is commutative. The keys to multiplication, then, are the same as that for simplifying – factor and then reduce.

Exercise #2: Simplify each of the following products.

(a) \( \frac{8y^7}{5x^3} \cdot \frac{10x^2}{6y^3} \)  
(b) \( \frac{6x^2y^3}{4x^3y^2} \cdot \frac{10xy^2}{9x^5y^7} \)

(c) \( \frac{2x^2 + 12x}{4x + 8} \cdot \frac{x^2 - 4x - 12}{x^2 - 36} \)  
(d) \( \frac{9 - x^2}{2x^3 - 6x^2} \cdot \frac{4x^2 - 4x}{x^2 + 2x - 3} \)
Division of rational expressions continues to follow from what you have seen in previous courses. Since division by a fraction can always be thought of in terms of multiplying by it reciprocal, these problems simply involve an additional step.

**Exercise #3:** Perform each of the following division problems. Express all answers in simplest form.

(a) \( \frac{15x^2}{6y^5} \div \frac{5x^8}{2y^7} \)

(b) \( \frac{30y^6}{20x^3} \div \frac{24y^2}{8x} \)

(c) \( \frac{x^2 + 2x - 8}{8x - 16} \div \frac{x^2 - 16}{2x + 10} \)

(d) \( \frac{9x^2 - 1}{3x^2 + 7x + 2} \div \frac{5 - 15x}{x^2 - 5x - 14} \)

**Exercise #4:** When \( \frac{x^2 - 25}{3x} \) is divided by \( \frac{x + 5}{9x} \) the result is

(1) \( \frac{x + 5}{27x} \)

(2) \( 3x - 15 \)

(3) \( \frac{x - 20}{3} \)

(4) \( 9x - 5 \)
**MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS**

**COMMON CORE ALGEBRA II HOMEWORK**

**SKILLS**

1. Express each of the following products in simplest form.

   (a) \( \frac{12x^4}{5y^8} \cdot \frac{15y}{30x^2} \)  
   (b) \( \frac{14a^2}{15b^9} \cdot \frac{10b^3}{21a^6} \)  
   (c) \( \frac{4x^3}{9z^5} \cdot \frac{3y^7}{10x^2} \cdot \frac{30z^2}{8y^3} \)

2. Write each of the following products in simplest form.

   (a) \( \frac{9x^2 - 16}{12x + 16} \cdot \frac{8x + 8}{3x^2 - x - 4} \)  
   (b) \( \frac{x^2 - x - 12}{x^2 + 8x + 15} \cdot \frac{x^2 + 2x - 15}{16 - x^2} \)

   (c) \( \frac{2x^2 + 7x - 4}{8x^3 - 4x^2} \cdot \frac{12x^2 - 24x}{x^2 + 6x + 8} \)  
   (d) \( \frac{x^2 - 7x - 8}{1 - x^2} \cdot \frac{3x^2 - 4x + 1}{9x^2 - 1} \)
3. When $\frac{24x^{10}}{2y^2}$ is divided by $\frac{36x^2}{6y^8}$ the result is

   (1) $2x^8y^7$  \hspace{1cm} (3) $\frac{x^8}{3y^7}$

   (2) $\frac{3x^5}{2y^7}$  \hspace{1cm} (4) $\frac{x^4}{2y^7}$

4. Express the result of each division problem below in simplest form.

   (a) $\frac{5x^3 - 10x^2}{10x^2 + 40x} \div \frac{x^2 - 5x + 6}{x^2 + x - 12}$

   (b) $\frac{24 - 18x}{9x^2 - 16} \div \frac{2x^2 + 2x}{3x^2 + 7x + 4}$

   (c) $\frac{x^2 - 6x + 8}{3x^4 - 6x^3} \div \frac{4x^2 - 1}{2x^3 - x^2}$

   (d) $\frac{49 - x^2}{x^2 - 9x + 14} \div \frac{x^2 + 2x - 35}{6 - 3x}$
COMMON CORE ALGEBRA II, UNIT #10 – POLYNOMIAL AND RATIONAL FUNCTIONS – LESSON #8

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COMBINING RATIONAL EXPRESSIONS WITH ADDITION AND SUBTRACTION
COMMON CORE ALGEBRA II

Occasionally it will be important to be able to combine two or more rational expressions by addition. Keep in mind two key principles that dictate fraction addition.

TWO GUIDELINES FOR ADDITION AND SUBTRACTION OF FRACTIONS

1. Fractions must have a common denominator. 2. Denominators can only be changed by multiplying the overall fraction by one.

Exercise #1: Combine each of the following fractions by first finding a common denominator. Express your answers in simplest form.

(a) \( \frac{2x-5}{4x} + \frac{4x+2}{6x} \)  
(b) \( \frac{4x-1}{5x} + \frac{x+5}{10} \)  
(c) \( \frac{3}{4x} + \frac{1}{2x^2} \)

Each of the combinations in Exercise #1 should have been reasonably easy because each denominator was monomial in nature. If this is not the case, then it is wise to factor the denominators before trying to find a common denominator.

Exercise #2: Combine each of the following fractions by factoring the denominators first. Then find a common denominator and add.

(a) \( \frac{4}{5y-15} + \frac{5}{y^2-9} \)  
(b) \( \frac{x-3}{x^2-9x+20} + \frac{2}{x^2-6x+8} \)
Subtraction of rational expressions is especially challenging because of errors that naturally arise when students forget to distribute the subtraction (or the multiplication by -1). Still, with careful execution, these problems are no different than their addition counterparts.

**Exercise #3:** Perform each of the following subtraction problems. Express your answers in simplest form.

(a) \[ \frac{3x + 7}{x^2 - 4} - \frac{x + 3}{x^2 - 4} \]

(b) \[ \frac{x - 3}{4x^2 - 1} - \frac{2}{10x + 5} \]

(c) \[ \frac{x}{x^2 - 4} - \frac{6}{x^2 + 8x - 20} \]

(d) \[ \frac{x - 2}{x^2 + 5x + 4} - \frac{8}{x^2 + 12x + 32} \]

**Exercise #4:** Which of the following is equivalent to \( \frac{1}{x - 1} - \frac{1}{x} \)?

(1) \( \frac{x}{x - 1} \)  
(2) \( \frac{1}{x - x^2} \)
(3) \( \frac{1}{x^2 - x} \)  
(4) \( \frac{x}{x^2 - 1} \)
COMBINING RATIONAL EXPRESSIONS WITH ADDITION AND SUBTRACTION
COMMON CORE ALGEBRA II HOMEWORK

**FLUENCY**

1. Combine each of the following using addition. Simply your result whenever possible.
   
   (a) \( \frac{3x-1}{6} + \frac{2x+5}{9} \)  
   (b) \( \frac{x}{10} + \frac{1}{15x} \)  
   (c) \( \frac{3}{7x} + \frac{5}{14x^2} \)

2. Combine and simplify each of the following. Note that each pair of fractions already has a common denominator.
   
   (a) \( \frac{3x+7}{x+2} + \frac{2x+3}{x+2} \)  
   (b) \( \frac{5x+2}{4x-12} - \frac{3x+8}{4x-12} \)  
   (c) \( \frac{6x^2 - 8x}{x^2 - 25} - \frac{4x^2 + 2x}{x^2 - 25} \)

3. Combine each of the following using addition. Simplify your final answers.
   
   (a) \( \frac{x}{5x+25} + \frac{2x-3}{x^2-3x-40} \)  
   (b) \( \frac{x-4}{x^2-24x+128} + \frac{2}{x^2-12x+32} \)
4. Which of the following represents the sum of \( \frac{1}{x+1} \) and \( \frac{1}{x-1} \)?

(1) \( \frac{2x}{x^2-1} \)  
(2) \( \frac{1}{x} \)  
(3) \( \frac{2}{x-1} \)  
(4) \( \frac{2x}{x^2+1} \)

5. When the expressions \( \frac{x^2-8x}{9-x^2} \) and \( \frac{3x+6}{9-x^2} \) are added the result can be written as

(1) \( \frac{x-5}{x-3} \)  
(2) \( \frac{x+2}{x-3} \)  
(3) \( \frac{2-x}{x+3} \)  
(4) \( \frac{x+7}{x-3} \)

6. Express each of the following differences in simplest form.

(a) \( \frac{x+2}{x^2+4x-32} - \frac{4}{x^2-16} \)
(b) \( \frac{2x+3}{8x^2+6x+1} - \frac{3}{2x^2-x-1} \)

7. When \( \frac{7x+14}{3x+12} \) is subtracted from \( \frac{2x-6}{3x+12} \) the result can be simplified to

(1) \( \frac{-5}{3} \)  
(2) \( \frac{-2}{3} \)  
(3) \( \frac{10}{3} \)  
(4) \( \frac{7}{3} \)
COMMON CORE ALGEBRA II

Complex fractions are simply defined as fractions that have other fractions within their numerators and/or denominators. To simplify these fractions means to remove these minor fractions and then eliminate any common factors. The key, as always, is to multiply by the number one in ways that simplify the fraction.

**Exercise #1:** Consider the complex fraction \( \frac{\frac{1}{9} + \frac{18}{1}}{\frac{3}{1}} \).

(a) What is the least common denominator amongst the three minor fractions?

(b) Multiply the numerator and denominator of the major fraction by your answer in part (a) and then simplify your result.

(c) Why is it acceptable to perform the operation in part (b)? What number are you effectively multiplying by?

By multiplying the major fraction by the number one, by using the least common denominator, we will always eliminate the minor fractions (by turning them into integer expressions).

**Exercise #2:** Simplify each of the following complex fractions.

\[
\begin{align*}
\text{(a) } \frac{1}{2} - \frac{1}{10} & \quad \text{(b) } \frac{2}{3} + \frac{2}{x} & \quad \text{(c) } \frac{3}{8} + \frac{1}{4x} \\
\frac{2}{5} & \quad \frac{5}{5} + \frac{5}{3} & \quad \frac{3}{7} + \frac{3}{2x} + \frac{3}{4}
\end{align*}
\]
These types of problems can certainly involve more complicated secondary simplification. Don’t forget the primary use of factoring in order to simplify.

**Exercise #3:** Simplify each of the following complex fractions.

(a) \[ \frac{1 - \frac{2}{x^2}}{\frac{2}{3} - \frac{3}{2x}} \]

(b) \[ \frac{\frac{2}{5}}{\frac{1}{x} - \frac{1}{x^2}} \]

(c) \[ \frac{\frac{x + 1}{12}}{\frac{3}{x} - \frac{2}{3x}} \]

If the denominators of the minor fractions become more complex, be sure to factor them first, just as you did with the addition and subtraction in the previous lesson.

**Exercise #4:** Simplify each of the following complex fractions.

(a) \[ \frac{4 + \frac{2}{x+2}}{\frac{x-4}{12x-24}} - \frac{\frac{2}{x-4}}{x^2-2x-8} \]

(b) \[ \frac{\frac{x}{x+6} - \frac{1}{x+2}}{\frac{x^2-4}{x^3+8x+12}} \]
COMMON CORE ALGEBRA II HOMEWORK

1. Simplify each of the following numerical complex fractions.

   (a) \( \frac{\frac{1}{4} + \frac{3}{20}}{\frac{1}{2}} \)
   (b) \( \frac{\frac{5}{18} + \frac{6}{1}}{\frac{1}{3}} \)
   (c) \( \frac{\frac{3}{4} - \frac{1}{5}}{\frac{1}{4}} \)

2. Simplify each of the following complex fractions.

   (a) \( \frac{\frac{1}{2} + \frac{1}{3x}}{\frac{3}{10} + \frac{1}{5x}} \)
   (b) \( \frac{2 - \frac{1}{2x}}{\frac{5}{1} + \frac{1}{x}} \)
   (c) \( \frac{\frac{1}{8} - \frac{2x}{1}}{\frac{1}{12x} - \frac{1}{3x^2}} \)

3. Simplify each of the following complex fractions.

   (a) \( \frac{\frac{5}{3x} - \frac{5}{x^2}}{\frac{1}{3} - \frac{3}{x^2}} \)
   (b) \( \frac{\frac{x}{10} - \frac{1 - 2}{x}}{\frac{1}{2} - \frac{x}{10}} \)
   (c) \( \frac{\frac{3}{4x} - \frac{3}{2}}{\frac{1}{8x^2}} \)
4. Simplify each of the following complex fractions.

\[
\begin{align*}
&\text{(a)} & \frac{x}{x-4} + \frac{4}{x-10} & \frac{3x+2}{x-1} - \frac{8}{x-4} \\
& & \frac{5x+10}{x^2-14x+40} & \frac{2x^2-12x}{x^2-5x+4} \\
\end{align*}
\]

5. Which of the following is equivalent to \( \frac{1}{x-1} - \frac{1}{x} \)?

\[
\begin{align*}
&\text{(1) 1} & \text{(3) } \frac{x}{x-1} \\
&\text{(2) } \frac{2}{x-1} & \text{(4) } x - x^2 \\
\end{align*}
\]

**Reasoning**

6. Since one can multiply by the number 1 at any point in an expression, simplify the following complex fraction by simplifying the more minor complex fraction first, then continue

\[
\begin{align*}
&\frac{1}{2} - \frac{1}{x} \\
&\frac{1}{10x} - \frac{1}{x} \\
&\frac{x}{2} - \frac{1}{10x} \\
&\frac{1}{2} - \frac{1}{5x^2}
\end{align*}
\]
POLYNOMIAL LONG DIVISION
COMMON CORE ALGEBRA II

We have worked to simplify rational expressions (polynomials divided by polynomials). In this lesson, we will look more closely at the division of two polynomials and how it is analogous to the division of two integers.

Exercise #1: Consider the division problem 1519 \div 7, which could also be written as \( \frac{1519}{7} \) and \( 7\overline{1519} \).

(a) Find the result of this division using the standard long division algorithm. Is there a remainder in this division?

(b) Rewrite your result from (a) as an equivalent multiplication equation.

(c) Now evaluate \( \frac{1522}{7} \) using long division. Write your answer in \( a + Rb \) form and in \( a + \frac{b}{c} \) form.

(d) Write your answer from part (c) as an equivalent multiplication equation.

Exercise #2: Now let's see how this works out when we divide two polynomials.

(a) Simplify \( \frac{2x^2 + 15x + 18}{x + 6} \) by performing polynomial long division.

(b) Rewrite the result you found in (a) as an equivalent multiplication equation.

(c) Write \( \frac{2x^2 + 15x + 20}{x + 6} \) in the form \( q(x) + \frac{r}{x + 6} \), where \( q(x) \) is a polynomial and \( r \) is a constant, by performing polynomial long division. Also, write the result an equivalent multiplication equation.
So, when we divide two polynomials, we always get another polynomial and a remainder. This is known as writing the rational expression in **quotient-remainder form**.

**Exercise #3:** Write each of the following rational expressions in the form \( q(x) + \frac{r}{x-a} \) form.

(a) \( \frac{x^2 + 2x - 5}{x - 3} \) \hspace{1cm} (b) \( \frac{2x^2 - 23x + 17}{x - 10} \)

Sometimes we can use the structure of an expression instead of polynomial long division.

**Exercise #4:** Consider the expression \( \frac{x + 8}{x + 3} \). We would like to write this as \( a + \frac{b}{x + 3} \).

(a) Write the numerator as an equivalent expression involving the expression \( x + 3 \). \hspace{1cm} (b) Use the fact that division distributes over addition to write the final answer.

We can extend what we did in the last problem to more challenging structure problems.

**Exercise #5:** Write each of the following in the form of \( a + \frac{b}{x - r} \).

(a) \( \frac{4x + 13}{x + 2} \) \hspace{1cm} (b) \( \frac{3x - 5}{x - 4} \)
FLUENCY

1. Write each of the following rational expressions in the form \( a + \frac{r}{x-b} \). Do these by rewriting your numerator as was done in Exercises #4 and #5.

   (a) \( \frac{x+6}{x+2} \)  
   (b) \( \frac{x-10}{x-3} \)

   (c) \( \frac{2x+5}{x+2} \)  
   (d) \( \frac{5x-2}{x-4} \)

2. If the expression \( \frac{10x+11}{2x+1} \) was placed in the form \( 5 + \frac{a}{2x+1} \), then which of the following would be the value of \( a \)?
   (1) 6  
   (2) −7  
   (3) 3  
   (4) −5

3. Use polynomial long division to simplify each of the following ratios. There should be a zero remainder.

   (a) \( \frac{x^2 + 5x - 24}{x - 3} \)  
   (b) \( \frac{6x^2 + 11x - 10}{3x - 2} \)
4. Use polynomial long division to write each of the following ratios in \( q(x) + \frac{r}{x-a} \) form, where \( q(x) \) is a polynomial and \( r \) is the remainder.

(a) \( \frac{x^2 - 6x + 11}{x - 4} \) 

(b) \( \frac{x^2 + 2x - 25}{x + 7} \)

(c) \( \frac{3x^2 + 17x + 25}{x + 4} \) 

(d) \( \frac{5x^2 - 41x + 3}{x - 8} \)

5. Write each of the following in \( q(x) + \frac{r}{x-a} \). The polynomial \( q(x) \) will now be a quadratic.

(a) \( \frac{x^3 + 7x^2 + 17x + 41}{x + 5} \) 

(b) \( \frac{2x^3 - 11x^2 + 22x - 25}{x - 3} \)
THE REMAINDER THEOREM
COMMON CORE ALGEBRA II

In the last lesson, we saw how two polynomials, when divided, resulted in another polynomial and a remainder. The remainder has a remarkable property in certain types of division. We will explore this relationship in the first exercise.

Exercise #1: Consider each of the following scenarios where we have \( \frac{p(x)}{x-a} \). In each case, simplify the division using polynomial long division and then evaluate \( p(a) \).

(a) \( \frac{x^2 - 8x + 18}{x - 2} \)
   \[ p(x) = x^2 - 8x + 18 \Rightarrow p(2) = \]

(b) \( \frac{x^2 - 2x - 25}{x - 7} \)
   \[ p(x) = x^2 - 2x - 25 \Rightarrow p(7) = \]

(c) \( \frac{2x^2 + 11x + 11}{x + 3} \)
   \[ p(x) = 2x^2 + 11x + 11 \Rightarrow p(-3) = \]

(d) \( \frac{3x^2 + 7x - 20}{x + 4} \)
   \[ p(x) = 3x^2 + 7x - 20 \Rightarrow p(-4) = \]
**The Remainder Theorem**

When the polynomial $p(x)$ is divided by the linear factor $(x-a)$ then the remainder will always be $p(a)$. In other words:

$$\frac{p(x)}{x-a} = q(x) + \frac{r}{x-a}$$

**Exercise #2:** If the ratio $\frac{x^2-11x+22}{x-9}$ was placed in the form $q(x) + \frac{r}{x-9}$, where $q(x)$ is a linear function, then which of the following is the value of $r$?

1. $-3$
2. $5$
3. $-9$
4. $4$

In the past, the remainder theorem was used primarily to aid in evaluating polynomials. These days it is the primary justification for telling if a linear expression is a factor of a polynomial.

**Exercise #3:** By definition $(x-a)$ is a factor of $p(x)$ if $\frac{p(x)}{x-a} = q(x)$, where $q(x)$ is another polynomial.

What must be true of the remainder, $p(a)$, for $(x-a)$ to be a factor of $p(x)$? Explain.

**Exercise #4:** Determine if each of the following are factors of the listed polynomials by evaluating the polynomials.

(a) Is $x-3$ a factor of $p(x) = x^2 - 11x + 24$? 
(b) Is $x+5$ a factor of $p(x) = 2x^2 + 9x - 2$?

(d) Is $x+1$ a factor of $p(x) = x^3 - 7x^2 - 11x - 3$ 
(c) Is $x-5$ a factor of $p(x) = x^3 - x^2 - 19x - 10$?

**Exercise #5:** For what value of $k$ will $x-4$ be a factor of $x^2 + kx - 52$? Show how you arrived at your answer.
THE REMAINDER THEOREM
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is the remainder when the polynomial \( x^2 - 5x + 3 \) is divided by the binomial \( (x - 8) \)?

   (1) 107          (3) 3
   (2) 27           (4) 9

2. If the ratio \( \frac{2x^2 + 17x + 42}{x + 5} \) is placed in the form \( q(x) + \frac{r}{x + 5} \), where \( q(x) \) is a polynomial, then which of the following is the correct value of \( r \)?

   (1) -3           (3) 18
   (2) 177          (4) 7

3. When the polynomial \( p(x) \) was divided by the factor \( x - 7 \) the result was \( x + \frac{11}{x - 7} \). Which of the following is the value of \( p(7) \)?

   (1) -8           (3) 11
   (2) 7            (4) It does not exist

4. Which of the following binomials is a factor of the quadratic \( 4x^2 - 35x + 24 \)? Try to do this without factoring but by using the Remainder Theorem.

   (1) \( x + 6 \)   (3) \( x - 8 \)
   (2) \( x - 4 \)   (4) \( x + 2 \)

5. Which of the following linear expressions is a factor of the cubic polynomial \( x^3 + 9x^2 + 16x - 12 \)?

   (1) \( x + 6 \)          (3) \( x - 3 \)
   (2) \( x - 1 \)          (4) \( x + 2 \)
6. Consider the cubic polynomial \( p(x) = x^3 + x^2 - 46x + 80 \).

(a) Using polynomial long division, write the ratio of \( \frac{p(x)}{x - 3} \) in **quotient-remainder form**, i.e. in the form \( q(x) + \frac{r}{x - 3} \). Evaluate \( p(3) \). How does this help you check your quotient-remainder form?

(b) Evaluate \( p(5) \). What does this tell you about the binomial \( x - 5 \)?

(c) If \( q(x) = \frac{p(x)}{x - 5} \), then use polynomial long division to find an expression for the polynomial \( q(x) \).

(d) Use your answer from (c) to **completely factor** the cubic polynomial \( p(x) \). Besides \( x = 5 \), what are its other zeroes?

7. For the cubic \( x^3 + 7x^2 + 13x + 3 \) has only one rational zero, \( x = -3 \). Use polynomial long division to show that the remainder is zero when dividing the cubic by \( x + 3 \). Then use the quadratic formula to find the other two (irrational) zeroes.
SOLVING FRACTIONAL EQUATIONS
COMMON CORE ALGEBRA II

Equations involving fractions or rational expressions arise frequently in mathematics. The key to working with them is to manipulate the equation, typically by multiplying both sides of it by some quantity, that eliminates the fractional nature of the equation. The most common form of this practice is “cross-multiplying.”

**Exercise #1:** Use the technique of cross multiplication to solve each of the following equations.

\[
\begin{align*}
(a) \quad \frac{4x + 5}{2} &= \frac{x - 1}{5} \\
(b) \quad \frac{x + 1}{2-x} &= \frac{2}{x-6}
\end{align*}
\]

Since this technique should be familiar to students at this point, we will move onto a less familiar method when more than two fractions are involved. The next exercise will illustrate the process.

**Exercise #2:** Consider the equation \( \frac{1}{2} - \frac{9}{4x} = \frac{3}{4x} \).

(a) What is the least common denominator for all three fractions in this equation?

(b) Multiply both sides of this equation by the LCD to “clear” the equation of the denominators. Now, solve the resulting linear equation.

It is very important to note the similarities and differences between this technique and the one employed to simplify complex fractions. With complex fractions we multiplied by one in creative ways. Here we are multiplying both sides of an equation by a quantity that removes the fractional nature of the equation.

**Exercise #3:** Which of the following values of \( x \) solves: \( \frac{x-4}{6} + \frac{x-2}{10} = \frac{31}{15} \)?

(1) \( x = 14 \)  
(2) \( x = 6 \)  
(3) \( x = -8 \)  
(4) \( x = 11 \)
These equations can involve quadratic as well as root expressions. The key, though, remains the same – multiplying both sides of the equation by the same quantity.

**Exercise #4:** Solve each of the following equations for all values of \( x \).

(a) \[ \frac{1}{10} - \frac{1}{x} = \frac{1}{5x} - \frac{2}{x^2} \]

(b) \[ \frac{1}{2} + \frac{3}{x} - \frac{1}{x^2} = \frac{1}{4x} + \frac{1}{2x^2} \]

Because fractional equations often involve denominators containing variables, it is important that we check to see if any solutions to the equation make it undefined. These represent further examples of extraneous roots.

**Exercise #5:** Solve and reject any extraneous roots.

(a) \[ \frac{x + 1}{x + 5} + \frac{18}{x^2 + 8x + 15} = \frac{9}{x + 3} \]

(b) \[ \frac{4}{x^2 + 4x - 12} + \frac{x - 1}{x + 6} = \frac{1}{x - 2} \]
SOLVING FRACTIONAL EQUATIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following fractional equations. After “clearing” the denominators you should have a linear equation to solve.
   
   (a) \( \frac{x - 2}{3} + \frac{x + 1}{6} = \frac{3}{2} \)
   
   (b) \( \frac{13}{2x} - \frac{4}{15} = \frac{31}{6x} \)
   
   (c) \( \frac{5}{x + 2} + \frac{1}{2} = 3 \)

2. Solve each of the fractional equations for all value(s) of \( x \).
   
   (a) \( x - 8 = -\frac{12}{x} \)
   
   (b) \( \frac{3}{4} + \frac{1}{2x} = \frac{1}{2x} + \frac{1}{3x^2} \)

   (c) \( \frac{17}{x} - \frac{11}{x + 3} = \frac{5x + 8}{x + 3} \)

   (d) \( \frac{x + 10}{2} - \frac{13}{x + 1} = \frac{11}{3} \)
3. Solve the following equation for all values of $x$. Express your answers in simplest $a + bi$ form.

$$\frac{x}{9} = \frac{x - 3}{x - 1}$$

4. Solve the following equation for all values of $x$. Be sure to check for extraneous roots.

$$\frac{x}{\sqrt{x+11}} - 1 = \frac{1}{\sqrt{x+11}}$$

5. Solve each of the following equations. Be sure to check for extraneous roots.

(a) $$\frac{x+1}{x-5} + \frac{2}{x-6} = \frac{2}{x^2 -11x + 30}$$

(b) $$\frac{x-3}{x-7} - \frac{1}{x} = \frac{28}{x^2 -7x}$$
SOLVING RATIONAL INEQUALITIES
COMMON CORE ALGEBRA II

We have already seen the solving of inequalities including quadratic expressions. Rational inequalities, those that include algebraic fractions with variables in both their numerator and denominator, are important and pose an interesting challenge compared with quadratics. The first exercise will illustrate the thinking involved in finding the solution set to a rational inequality.

**Exercise #1:** Consider the rational inequality \( \frac{x - 5}{x + 3} \geq 0 \).

(a) At what \( x \)-value is the ratio \( \frac{x - 5}{x + 3} \) equal to zero?  (b) At what \( x \)-value is the ratio \( \frac{x - 5}{x + 3} \) undefined?  Is this value part of the solution set?

(c) Enter the ratio \( \frac{x - 5}{x + 3} \) in your calculator to help determine values of \( x \) that solve this inequality.  Plot its solution on a number line and state the answer in set-builder notation.

The key to solving rational inequalities that are compared to zero is to find the values of \( x \) that make the numerator or denominator equal to zero. These are known as the critical values of the rational expression. The rational expression can only change signs at these critical values.

**Exercise #2:** Solve the rational inequality \( \frac{x^2 + 3x - 4}{x^2 - 6x + 9} < 0 \) for all values of \( x \). Show your solution on a number line and state its solution in either interval or set-builder notation.
If a single algebraic ratio is compared to zero, then the solution method is fairly straightforward. It becomes more difficult if there exists more than one ratio or if the ratio is being compared to a quantity other than zero. In both cases, it is important to algebraically manipulate the expression so that we are comparing it to zero.

**Exercise #3:** Solve the rational inequality \( \frac{x}{x-8} \leq 5 \). Represent your answer using a number line and using set-builder or interval notation.

Some of these inequalities can test all of your key fraction abilities.

**Exercise #4:** Solve the rational inequality \( \frac{x-1}{x} + \frac{x+2}{x+3} \leq \frac{3}{2} \). Represent your answer using a number line and using any appropriate notation.
SOLVING RATIONAL INEQUALITIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

Solve each of the following rational inequalities. Show your answers using a number line and an appropriate notation.

1. \( \frac{x - 10}{x + 5} \geq 0 \)

2. \( \frac{2x + 1}{x + 3} < 0 \)

3. \( \frac{x^2 - 4}{x^2 - x - 20} > 0 \)

4. \( \frac{x^2 - 6x - 16}{x^2 - x - 6} \leq 0 \)

5. \( \frac{x^2 + 6x + 9}{4x^2 - 3x - 1} \geq 0 \)

6. \( \frac{x^2 - 12x + 36}{4x^2 - 4x + 1} < 0 \)
For problems 7 through 9, solve each rational inequality by first comparing it to zero. Represent your answers on a number line and using appropriate notation.

7. \( \frac{x + 1}{x - 3} \leq 2 \)

8. \( \frac{x^2 + 2x}{x + 4} > \frac{4}{3} \)

9. \( \frac{1}{x - 2} - \frac{1}{x + 2} \geq \frac{3}{x^2 - 4} \)
REASONING ABOUT RADICAL AND RATIONAL EQUATIONS
COMMON CORE ALGEBRA II

In previous lessons we have looked at solutions to square root (radical) and rational (fractional) equations. Sometimes, the solutions to these equations introduced extraneous roots that needed to be rejected. In this lesson we will look to justify the steps in solving these types of equations and understand why extraneous roots are introduced. First, let’s review two basic properties of equality.

**Properties of Equality**

1. **The Addition Property:** If \( a = b \) and \( c = d \) then \( a + c = b + d \).
2. **The Multiplication Property:** If \( a = b \) and \( c = d \) then \( a \cdot c = b \cdot d \).

Let’s use these two properties to justify the solution of an equation involving a square root.

**Exercise #1:** Give a reason or cite a property for each of the following lines in the solution of the square root equation \( x = \sqrt{x + 6} \).

\[
\begin{align*}
x &= \sqrt{x + 6} \\
x \cdot x &= \sqrt{x + 6} \cdot \sqrt{x + 6} \\
x^2 &= x + 6 \\
-x - 6 &= -x - 6 \\
x^2 - x - 6 &= 0 \\
(x - 3)(x + 2) &= 0 \\
x - 3 &= 0 \text{ or } x + 2 = 0 \\
x &= 3 \text{ or } x = -2
\end{align*}
\]

**Exercise #2:** Check each of the values of \( x \) from Exercise #1. Show your check. Which root is extraneous? Does the extraneous root satisfy the equation \( x^2 = x + 6 \)?

The extraneous root gets introduced because the operation of squaring is irreversible, meaning that once we’ve squared, we cannot know the original quantity. Let’s take a look at another exercise.

**Exercise #3:** Consider the relatively easy equation \( -2x = 6 \).

(a) Solve this equation by using the reversible operation of division.

(b) Solve this equation by first squaring both sides. What extraneous root has been introduced?
So, in the case of solving square root equations, because we square both sides at some point to “remove” the radical, we sometimes introduce solutions that do not make our original equation true. Let’s investigate why it happens sometimes with rational equations.

**Exercise #4:** Consider the rational equation \( \frac{x}{x-1} = \frac{2x+6}{x+3} \). Justify each step of the solution to this equation by citing a property or some other reason. Some steps of algebra have been omitted for the sake of space.

\[
\begin{align*}
(x-1)(x+3) \cdot \frac{x}{x-1} &= (x-1)(x+3) \cdot \frac{2x+6}{x+3} \\
 x(x+3) &= (x-1)(2x+6) \\
 x^2 + 3x &= 2x^2 + 4x - 6 \\
 -x^2 - 3x &= -x^2 - 3x \\
 0 &= x^2 + x - 6 \\
 0 &= (x+3)(x-2) \\
 x+3 &= 0 \text{ or } x-2 = 0 \\
 x = -3 \text{ or } x = 2
\end{align*}
\]

**Exercise #5:** Which of these two solutions is extraneous? How can you tell?

The **extraneous root** is arising in this case because multiplying both sides of an equation by anything containing a variable, like the factors \((x-1)\) and \((x+3)\), is **irreversible** due to the fact that we could be multiplying both sides by zero. Let’s investigate this by solving another equation.

**Exercise #6:** Consider the very simple equation \( 5x = 15 \) whose only solution is \( x = 3 \). Solve this equation by first multiplying both sides of this equation by \((x-1)\). What extraneous root has been introduced?
REASONING ABOUT RADICAL AND RATIONAL EQUATIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve the following equation involving a square root. Be sure to reject the extraneous solution (and there will be one).

\[ x = -\sqrt{3x + 10} \]

2. Solve the following rational equation. Reject any extraneous roots.

\[ \frac{x}{x - 3} + \frac{2}{x^2 - 7x + 12} = \frac{2}{x - 4} \]

REASONING

3. Consider the square root equation \( \sqrt{x} = x - 2 \).

(a) Show that \( x = 4 \) is a solution to this equation.

(b) The value \( x = 1 \) is not a solution to the original equation. Show that after squaring both sides, \( x = 1 \) is a solution to this new equation.
4. Given the equation $2x - 1 = 7$ answer the following.

(a) Solve this equation for the one and only value of $x$ that is a solution.

(b) What extraneous root is introduced if the first step taken to solve the equation is squaring both sides? Show the work that leads to this extraneous root.

5. Consider the equation $-4x = 12$, for which $x = -3$ is the only solution.

(a) If Dakota begins to solve the problem in the following way, what property could Dakota use to justify the unusual move of multiplying both sides by the expression $(x - 6)$?

\[
\begin{align*}
-4x(x - 6) &= 12(x - 6) \\
-4x^2 + 24x &= 12x - 72 \\
0 &= 4x^2 - 12x - 72 \\
0 &= x^2 - 3x - 18
\end{align*}
\]

(b) Solve the equation $0 = x^2 - 3x - 18$. What extraneous root was introduced by multiplying by $(x - 6)$ on both sides?

6. Squaring both sides of an equation is irreversibe. Is cubing both sides of an equation reversible? Provide numerical examples to help support your answer.
UNIT #11

THE CIRCULAR FUNCTIONS

Lesson #1 – Rotations and Angle Terminology
Lesson #2 – Radian Angle Measurement
Lesson #3 – The Unit Circle
Lesson #4 – The Definition of the Sine and Cosine Functions
Lesson #5 – More Work with the Sine and Cosine Functions
Lesson #6 – Basic Graphs of Sine and Cosine
Lesson #7 – Vertical Shifting of Sinusoidal Graphs
Lesson #8 – The Frequency and Period of a Sinusoidal Graph
Lesson #9 – Sinusoidal Modeling
Lesson #10 – The Tangent Function
Lesson #11 - The Reciprocal Functions
In this unit we will be studying the three basic trigonometric functions. These functions are based on the geometry of a circle and rotations around its center. Sometimes the trigonometric functions are known as circular functions. In this introductory lesson we introduce some basic terminology and concepts concerning angles. Some of the terminology is specified below.

**Standard Position:** An angle is said to be drawn in standard position if its vertex is at the origin and its initial ray points along the positive \(x\)-axis.

**Positive and Negative Rotations:** A rotation is said to be positive if the initial ray is rotated counter-clockwise to the terminal ray and said to be negative if the initial ray is rotated clockwise to the terminal ray.

**Coterminal Angles:** Any two angles drawn in standard position that share a terminal ray.

**Reference Angles:** The positive acute angle formed by the terminal ray and the \(x\)-axis.

**Exercise #1:** For each of the following angles, given by the Greek letter \(\theta\), draw a rotation diagram and identify the quadrant that the terminal ray falls in.

(a) \(\theta = 145^\circ\)  
(b) \(\theta = 320^\circ\)  
(c) \(\theta = 72^\circ\)  
(d) \(\theta = -210^\circ\)

(e) \(\theta = 250^\circ\)  
(f) \(\theta = -310^\circ\)  
(g) \(\theta = 460^\circ\)  
(h) \(\theta = -400^\circ\)
Exercise #2: In which quadrant would the terminal ray of an angle drawn in standard position fall if the angle measures 860°?

(1) I  (3) III  
(2) II  (4) IV  

Exercise #3: Give a negative angle that is coterminal with each of the following positive angles, \( \alpha \).

(a) \( \alpha = 90° \)  
(b) \( \alpha = 330° \)  
(c) \( \alpha = 120° \)  
(d) \( \alpha = 210° \)  

Exercise #4: Coterminal angles drawn in standard position will always have measures that differ by an integer multiple of

(1) 90°  
(2) 360°  
(3) 180°  
(4) 720°  

Exercise #5: For each of the following angles, \( \beta \), draw a rotation diagram and then state \( \beta \)'s reference angle, \( \beta_r \).

(a) \( \beta = 160° \)  
(b) \( \beta = 300° \)  
(c) \( \beta = 210° \)  
(d) \( \beta = 78° \)  

(e) \( \beta = -110° \)  
(f) \( \beta = -280° \)  
(g) \( \beta = 605° \)  
(h) \( \beta = -410° \)
ROTATIONS AND ANGLE TERMINOLOGY
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. For each of the following angles, draw a rotation diagram and then state the quadrant the terminal ray of the angles falls within.

   (a) $\theta = 135^\circ$
   (b) $\theta = 300^\circ$
   (c) $\theta = -110^\circ$

   (d) $\theta = -310^\circ$
   (e) $\theta = 85^\circ$
   (f) $\theta = 560^\circ$

2. For each of the following angles, draw a rotation diagram and determine the reference angle.

   (a) $\alpha = 245^\circ$
   (b) $\alpha = 290^\circ$
   (c) $\alpha = 130^\circ$

   (d) $\alpha = -242^\circ$
   (e) $\alpha = 475^\circ$
   (f) $\alpha = -432^\circ$
3. Give two angles that are coterminal with each of the following angles. Make one of the coterminal angles positive and one negative.

(a) $\theta = 105^\circ$    (b) $\theta = 220^\circ$    (c) $\theta = 80^\circ$    (d) $\theta = -245^\circ$

4. When drawn in standard position, which of the following angles is coterminal to one that measures $130^\circ$?

(1) $430^\circ$    (3) $850^\circ$

(2) $-70^\circ$    (4) $730^\circ$

5. Which of the following angles, when drawn in standard position, would not be coterminal with an angle that measures $270^\circ$?

(1) $-90^\circ$    (3) $630^\circ$

(2) $990^\circ$    (4) $720^\circ$

6. Which of the following angles would not have a reference angle equal to $30^\circ$?

(1) $210^\circ$    (3) $120^\circ$

(2) $-330^\circ$    (4) $-30^\circ$

REASONING

7. Angles are a measurement of rotation about a point. Are two coterminal angles the same rotation? Explain your answer. Diagrams are helpful.
Radian Angle Measurement
Common Core Algebra II

Just as distance can be measured in inches, feet, miles, centimeters, etcetera, rotations about a point can also be measured in many different ways. Measuring one complete rotation in terms of $360^\circ$ is somewhat arbitrary. A common unit of angle measurement that is an alternative to degrees is called the radian. You first saw this unit of angle in Common Core Geometry. It is natural to define it in terms of the geometry of a circle, given the unit we are in:

**The Definition of a Radian**

The radian angle, $\theta$, created by a rotation about a point $A$ using a radius of $r$ and passing through an arc length of $s$ is defined as

$$\theta = \frac{s}{r}$$

or equivalently

$$s = \theta \cdot r$$

**Exercise #1:** Consider a full rotation around any point in the counter-clockwise (positive) direction.

(a) What is the arc length, $s$, in terms of the radius of the circle, $r$, for a full rotation?

(b) Based on the definition above and on your answer to part (a), how many radians are there in one full rotation?

Radians essentially measure the total number of radii (hence the name) that have been traversed about the circumference of a circle in a given rotation. Based on the circumference formula of a circle, thus, there will always be $2\pi$ radians in one full rotation.

**Exercise #2:** Use the formula above to answer each of the following.

(a) Determine the number of radians that the minute hand of a clock passes through if it has a length of 5 inches and its tip travels a total distance of 13 inches.

(b) If a pendulum swings through an angle of 0.55 radians, what distance does its tip travel if it has a length of 8 feet?
Radian are essential in the study of higher-level mathematics and physics and are the angle measurement of choice for the study of calculus. It is important to be able to convert between the angular system of degrees and that of radians. The next exercise will illustrate this process.

**Exercise #3:** Consider one-half of a full rotation.

(a) What is the angle of rotation in both degrees and in radians? 
(b) Using the two equivalent angles of rotation in (a), convert a 30° angle into an equivalent angle in radians.

**Exercise #4:** Convert each of the following common angles in degrees into radians. Express your answers in terms of pi.

(a) \( \theta = 90^\circ \)  
(b) \( \theta = 120^\circ \)  
(c) \( \theta = 225^\circ \)

**Exercise #5:** Convert each of the following common radian angles into degrees.

(a) \( \theta = \frac{5\pi}{6} \)  
(b) \( \theta = \frac{3\pi}{2} \)  
(c) \( \theta = \frac{3\pi}{4} \)

We should also feel comfortable with the fact that radians do not always have to be in terms of pi, although they often are.

**Exercise #6:** Convert each of the following radian angles, which aren’t in terms of pi, into degrees. Round your answers to the nearest degree.

(a) \( \theta = 5.8 \)  
(b) \( \theta = 4.2 \)  
(c) \( \theta = -2.5 \)

**Exercise #7:** An angle drawn in standard position whose radian measure is 2 radians would terminate in which of the following quadrants?

(1) I  
(2) II  
(3) III  
(4) IV
Radian Angle Measurement

Common Core Algebra II Homework

Fluency

1. Convert each of the following common degree angles to angles in radians. Express your answers in exact terms of pi.
   (a) 30°    (b) 45°    (c) 60°    (d) 180°
   (e) 300°    (f) 135°    (g) 270°    (h) 330°

2. Convert each of the following angles given in radians into an equivalent measure in degrees. Your answers will be integers.
   (a) $\frac{2\pi}{3}$    (b) $-\frac{\pi}{2}$    (c) $\frac{11\pi}{4}$    (d) $-\frac{4\pi}{3}$

3. If an angle is drawn in standard position with each of the following radians angles, determine the quadrant its terminal ray lies in. Hint – convert each angle into degrees.
   (a) 4.75    (b) -5.28    (c) 1.65    (d) 7.38
4. Draw a rotation diagram for each of the following radian angles, which are expressed in terms of \( \pi \). Then, determine the reference angle for each, also in terms of \( \pi \). Think back to how you did this with degrees.

(a) \( \frac{2\pi}{3} \)  
(b) \( \frac{11\pi}{6} \)  
(c) \( \frac{5\pi}{4} \)

APPLICATIONS

5. A dog is attached to a 10 foot leash. He travels around an arc that has a length of 25 feet. Which of the following represents the radian angle he has rotated through?

(1) 5  
(2) 7.5\( \pi \)  
(3) 2.5  
(4) 1.25\( \pi \)

6. A wheel whose diameter is 3 feet rolls a distance of 45 feet without slipping. Through what radian angle did the wheel rotate?

(1) 30  
(2) 25  
(3) 30\( \pi \)  
(4) 12\( \pi \)

7. The distance from the center of a Ferris wheel to a person who is riding is 38 feet. What distance does a person travel if the Ferris wheel rotates through an angle of 4.25 radians?

(1) 80.75 feet  
(2) 42.5 feet  
(3) 507 feet  
(4) 161.5 feet

8. A golfer swings a club about a pivot point. If the head of the club travels a distance of 26 feet and rotates through an angle of 5 radians, which of the following gives the distance the club head is from the pivot point?

(1) 1.7 feet  
(2) 2.6 feet  
(3) 5.2 feet  
(4) 7.2 feet
The basis of trigonometry will be a very special circle known as the unit circle. This is simply a circle that has its center located at the origin and has a radius equal to one unit (hence the name "unit").

**Exercise #1:** From our work with equations of circles, which of the following would represent the equation of the unit circle?

1. \( x + y = 1 \)
2. \( y = x^2 + 1 \)
3. \( x^2 + y^2 = 1 \)
4. \( (x-1)^2 + (y-1)^2 = 1 \)

Next we will seek to produce some of the coordinate points that lie on the unit circle through the use of the Pythagorean Theorem. The next two exercises will illustrate the important right triangles we will need.

**Exercise #2:** Consider the right triangle shown whose hypotenuse is equal to one and whose angles are both equal to 45°. Since this is an isosceles right triangle, the two equal sides are labeled \( x \). Solve for \( x \) and place your answer in simplest radical form.

**Exercise #3:** Consider the 30°-60° right triangle shown whose hypotenuse is equal to one. Clearly this triangle is half of an equilateral triangle.

(a) What is the length of the shorter side of this right triangle?

(b) Using the Pythagorean Theorem, find the length of the longer side in simplest radical form.
**Exercise #4:** The diagram below represents the **unit circle**. Based on your work from Exercises #2 and #3, fill in the ordered pairs at each of the following angles that are assumed to be drawn in standard position.

**Exercise #4:** For each of the following angles drawn in standard position, give the coordinate pair from the unit circle.

(a) $-120^\circ$  
(b) $495^\circ$  
(c) $\frac{\pi}{3}$  
(d) $\frac{3\pi}{2}$
THE UNIT CIRCLE
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Draw a rotation diagram for each of the following angles and then determine the ordered pair that lies on the unit circle for each angle.

   (a) $\theta = 330^{\circ}$
   (b) $\theta = 135^{\circ}$
   (c) $\theta = -270^{\circ}$

   (d) $\theta = -240^{\circ}$
   (e) $\theta = 540^{\circ}$
   (f) $\theta = -300^{\circ}$

2. Draw a rotation diagram for each of the following radian angles and then determine the ordered pair that lies on the unit circle for each angle.

   (a) $\alpha = \frac{2\pi}{3}$
   (b) $\alpha = -\frac{3\pi}{2}$
   (c) $\alpha = \frac{11\pi}{6}$

   (d) $\alpha = -\frac{\pi}{2}$
   (e) $\alpha = \frac{3\pi}{4}$
   (f) $\alpha = \frac{4\pi}{3}$
3. All of the points on the unit circle must satisfy the equation $x^2 + y^2 = 1$. Verify that this equation is true for each of the coordinate points given below.

(a) $(-1, 0)$  

(b) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$  

(c) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

4. There are other points on the unit circle besides the ones that we determined in this lesson. Every point, though, must satisfy the equation $x^2 + y^2 = 1$. For each of the following problems, either the $x$ or $y$ coordinate of a point on the unit circle is given. Find all possibilities for the other coordinate for this point using the unit circle equation.

(a) $x = \frac{3}{5}$  

(b) $y = -\frac{5}{13}$  

(c) $x = \frac{1}{4}$

5. For each of the following angles, determine its reference angle. Then state the coordinate on the unit circle for both the angle and its reference. What do you notice about the coordinate pairs?

(a) $\theta = 150^\circ$  

(b) $\theta = 225^\circ$  

(c) $\theta = 300^\circ$
THE DEFINITION OF THE SINE AND COSINE FUNCTIONS
COMMON CORE ALGEBRA II

The sine and cosine functions form the basis of trigonometry. We would like to define them so that their definition is consistent with what you already are familiar with concerning right triangle trigonometry. Recall from Common Core Geometry that in a right triangle the sine and cosine ratios were defined as:

\[
\sin A = \frac{\text{side length opposite of } A}{\text{length of the hypotenuse}} \quad \text{and} \quad \cos A = \frac{\text{side length adjacent to } A}{\text{length of the hypotenuse}}
\]

Exercise #1: Consider the unit circle shown below with an angle, \( \theta \), drawn in standard position.

(a) Given the right triangle shown, find an expression for \( \sin(\theta) \).

(b) Given the right triangle shown, find an expression for \( \cos(\theta) \).

We thus define the sine and cosine functions by using the coordinates on the unit circle. They are the first functions that are geometrically defined as they are based on the geometry of a circle (circular functions).

THE DEFINITION OF THE SINE AND COSINE FUNCTIONS

For an angle in standard position whose terminal ray passes through the point \( (x, y) \) on the unit circle:

\[
\sin(\theta) = \text{the } y\text{-coordinate} \quad \text{and} \quad \cos(\theta) = \text{the } x\text{-coordinate}
\]

The above definition is unquestionably the most important fact to memorize concerning trigonometry. We can now use this along with our work on the unit circle to determine certain exact values of cosine and sine.

Exercise #2: Using the unit circle diagram, determine each of the following values.

(a) \( \sin(30^\circ) = \) \hspace{1cm} (b) \( \sin(240^\circ) = \) \hspace{1cm} (c) \( \cos(90^\circ) = \) \hspace{1cm} (d) \( \cos(180^\circ) = \)

(e) \( \sin(90^\circ) = \) \hspace{1cm} (f) \( \sin(135^\circ) = \) \hspace{1cm} (g) \( \cos(150^\circ) = \) \hspace{1cm} (h) \( \cos(0^\circ) = \)
**Exercise #3:** The terminal ray of an angle, \( \alpha \), drawn in standard position passes through the point \((-0.6, 0.8)\), which lies on the unit circle. Which of the following gives the value of \( \sin(\alpha) \)?

1. 1.2  
2. 0.8  
3. -0.6  
4. 0.2

It is important to be able to determine the sign (positive or negative) of each of the two basic trigonometric functions for an angle whose terminal ray lies in a given quadrant. The next exercise illustrates this process.

**Exercise #4:** For each quadrant below, determine if the sine and cosine of an angle whose terminal ray falls in the quadrant is positive (+) or negative (−).

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin(\theta))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\cos(\theta))</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since each point on the unit circle must satisfy the equation \( x^2 + y^2 = 1 \), we can now state what is known as the **Pythagorean Identity**.

**The Pythagorean Identity**

For any angle, \( \theta \), \((\cos \theta)^2 + (\sin \theta)^2 = 1\)

**Exercise #5:** An angle, \( \alpha \), has a terminal ray that falls in the second quadrant. If it is known that \( \sin(\alpha) = \frac{3}{5} \), determine the value of \( \cos(\alpha) \).

**Exercise #6:** An angle, \( \theta \), has a terminal ray that falls in the first quadrant and \( \cos(\theta) = \frac{1}{3} \). Determine the value of \( \sin(\theta) \) in simplest radical form.
THE DEFINITION OF THE SINE AND COSINE FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is the value of \( \sin(60^\circ) \)?

   (1) \( \frac{\sqrt{2}}{2} \)  
   (2) \( \frac{1}{2} \)  
   (3) \( \frac{\sqrt{3}}{2} \)  
   (4) \( \frac{2}{3} \)

2. Written in exact form, \( \cos(135^\circ) = ? \)

   (1) \( \frac{1}{2} \)  
   (2) \( -\frac{\sqrt{2}}{2} \)  
   (3) \( -\frac{\sqrt{3}}{2} \)  
   (4) \( -\frac{\pi}{4} \)

3. Which of the following is not equal to \( \sin(270^\circ) \)?

   (1) \( \cos(180^\circ) \)  
   (2) \( -\cos(0^\circ) \)  
   (3) \( -\sin(90^\circ) \)  
   (4) \( \sin(360^\circ) \)

4. The terminal ray of an angle drawn in standard position passes through the point \( (0.28, -0.96) \), which lies on the unit circle. Which of the following represents the sine of this angle?

   (1) \( -0.96 \)  
   (2) \( -0.68 \)  
   (3) \( 0.28 \)  
   (4) \( -0.29 \)

5. The point \( A(-5, 12) \) lies on the circle whose equation is \( x^2 + y^2 = 169 \). Which of the following would represent the cosine of an angle drawn in standard position whose terminal rays passes through \( A \)?

   (1) \( -5 \)  
   (2) \( -\frac{5}{12} \)  
   (3) \( -\frac{5}{13} \)  
   (4) \( 12 \)
6. Which of the following values cannot be the sine of an angle? Hint, think about the range of \(y\)-values on the unit circle.

(1) \(\frac{7}{13}\)  
(2) \(-\frac{\sqrt{5}}{3}\)
(3) \(-\frac{3}{2}\)  
(4) \(\frac{\sqrt{11}}{4}\)

7. For an angle drawn in standard position, it is known that its cosine is negative and its sine is positive. The terminal ray of this angle must terminate in which quadrant?

(1) I  
(2) II  
(3) III  
(4) IV

8. If both the sine and cosine of an angle are less than zero, then when drawn in standard position in which quadrant would the terminal ray fall?

(1) I  
(2) II  
(3) III  
(4) IV

9. Which of the following has a cosine that is different from \(\sin(30^\circ)\)?

(1) 60°  
(2) −300°  
(3) −60°  
(4) 120°

10. When drawn in standard position, an angle \(\alpha\) has a terminal ray that lies in the second quadrant and whose sine is equal to \(\frac{9}{41}\). Find the cosine of \(\alpha\) in rational form (as a fraction).

11. If the terminal ray of \(\beta\) lies in the fourth quadrant and \(\sin(\beta) = -\frac{\sqrt{3}}{3}\) determine \(\cos(\beta)\) in simplest form.
THE UNIT CIRCLE

\[ \theta = 90^\circ, \frac{\pi}{2} \]

\((0, 1)\)

\[ \theta = 0^\circ, 360^\circ, 2\pi \]

\((-1, 0)\)

\[ \theta = 180^\circ, \pi \]

\[(\sqrt{2}/2, \sqrt{2}/2)\]

\[ \theta = 120^\circ, \frac{2\pi}{3} \]

\[ \theta = 60^\circ, \frac{\pi}{3} \]

\[ (\sqrt{2}/2, \sqrt{2}/2) \]

\[ \theta = 45^\circ, \frac{\pi}{4} \]

\[ (\sqrt{3}/2, 1/2) \]

\[ \theta = 30^\circ, \frac{\pi}{6} \]

\[ (-\sqrt{3}/2, 1/2) \]

\[ \theta = 150^\circ, \frac{5\pi}{6} \]

\[ (\sqrt{3}/2, -1/2) \]

\[ \theta = 330^\circ, \frac{11\pi}{6} \]

\[ (-\sqrt{2}/2, -\sqrt{2}/2) \]

\[ \theta = 225^\circ, \frac{5\pi}{4} \]

\[ (-\sqrt{2}/2, \sqrt{2}/2) \]

\[ \theta = 240^\circ, \frac{4\pi}{3} \]

\[ (-1/2, -\sqrt{3}/2) \]

\[ \theta = 300^\circ, \frac{5\pi}{3} \]

\[ (1/2, -\sqrt{3}/2) \]

\[ \theta = 360^\circ, 2\pi \]

\[ \theta = 270^\circ, \frac{3\pi}{2} \]
MORE WORK WITH THE SINE AND COSINE FUNCTIONS

COMMON CORE ALGEBRA II

The sine and cosine functions are the first a student typically encounters that are non-algebraic, that is they cannot be thought of as combinations of a finite number of integer powers and/or roots. Since they are defined by using the geometry of a circle they are not all that intuitive. Additional practice will be given in this lesson to simply get used to them.

Exercise #1: Recall the following definitions of the sine and cosine functions. If \( \theta \) is an angle drawn in standard position whose terminal ray passes through the point \((x, y)\) on the unit circle then …

\[
\sin(\theta) = \quad \text{and} \quad \cos(\theta) =
\]

Exercise #2: Given the function \( f(x) = 6\sin(x) \) which of the following is the value of \( f(60^\circ) \)?

(1) \(4\sqrt{2} \) \hspace{1cm} (3) \(3 \)
(2) \(3\sqrt{3} \) \hspace{1cm} (4) \(0 \)

Exercise #3: If \( g(\alpha) = 4\cos(\alpha) - 2\sin(\alpha) \) then \( g(330^\circ) = ? \)

(1) \(8\sqrt{2} + 3 \) \hspace{1cm} (3) \(2\sqrt{3} + 1 \)
(2) \(4\sqrt{3} - 2 \) \hspace{1cm} (4) \(6\sqrt{2} - 4 \)

Exercise #4: Which of the following is not equal to one?

(1) \(\cos(0^\circ) \) \hspace{1cm} (3) \(\sin(90^\circ) \)
(2) \(\cos(360^\circ) \) \hspace{1cm} (4) \(\sin(270^\circ) \)

Exercise #5: For an angle \( \alpha \) whose terminal ray lies in the third quadrant it is known that \( \cos(\alpha) = -0.96 \). Which of the following is the value of \( \sin(\alpha) \)?

(1) \(-0.28 \) \hspace{1cm} (3) \(0.04 \)
(2) \(-0.56 \) \hspace{1cm} (4) \(0.78 \)
A special relationship exists between the trigonometric values of an angle and those of its reference angle. This important relationship is illustrated in the next exercise.

**Exercise #6:** In each of the following, an angle and its reference have been given. Using your calculator, in **degree mode**, determine the sine and cosine of both the angle and its reference. Round all answers to the nearest **hundredth**.

(a) $\theta = 110^\circ$ and $\theta_r = 70^\circ$  
(b) $\theta = 235^\circ$ and $\theta_r = 55^\circ$  
(c) $\theta = 282^\circ$ and $\theta_r = 78^\circ$

Clearly the absolute value of the sine and cosine are the same for an angle and its reference. This fact can be exploited to produce sine and cosine values for angles if they are known for their references.

**Exercise #7:** Given that $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ and $\sin(30^\circ) = \frac{1}{2}$, determine the following values in exact form.

(a) $\cos(150^\circ)$  
(b) $\sin(150^\circ)$  
(c) $\cos(210^\circ)$

(d) $\sin(210^\circ)$  
(e) $\cos(330^\circ)$  
(f) $\sin(330^\circ)$

We should not forget that the trigonometric functions are valid for radians as well as degrees. Practice evaluating these functions for each of the following radian inputs. If needed, convert to degrees.

**Exercise #8:** Evaluate each of the following trigonometric expressions.

(a) $\sin\left(\frac{\pi}{2}\right)$  
(b) $\sin\left(\frac{\pi}{3}\right)$  
(c) $\cos\left(\frac{3\pi}{2}\right)$  
(d) $\cos\left(\frac{3\pi}{4}\right)$
MORE WORK WITH THE SINE AND COSINE FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. If \( f(x) = 10 \sin(x) - 3 \) then \( f(30^\circ) = ? \)

   \begin{align*}
   (1) & \quad -\frac{\sqrt{3}}{2} - 3 \\
   (2) & \quad 2 \\
   (3) & \quad -\frac{5}{2} \\
   (4) & \quad \frac{4}{3} - \frac{\sqrt{3}}{2}
   \end{align*}

2. If \( f(x) = 2x \) and \( g(x) = \cos(x) \) then \( g\left(f\left(\frac{\pi}{2}\right)\right) = ? \)

   \begin{align*}
   (1) & \quad 1 \\
   (2) & \quad -\frac{\sqrt{2}}{2} \\
   (3) & \quad 0 \\
   (4) & \quad -1
   \end{align*}

3. Which of the following represents a rational number?

   \begin{align*}
   (1) & \quad \sin\left(\frac{\pi}{6}\right) \\
   (2) & \quad \sin\left(\frac{2\pi}{3}\right) \\
   (3) & \quad \cos\left(\frac{\pi}{4}\right) \\
   (4) & \quad \cos\left(\frac{5\pi}{4}\right)
   \end{align*}

4. When drawn in standard position, an angle \( \beta \) has a terminal ray that lies in the third quadrant. It is known that \( \cos(\beta) = -\frac{8}{17} \). Which of the following represents the value of \( \sin(\beta) \)?

   \begin{align*}
   (1) & \quad -\frac{9}{17} \\
   (2) & \quad \frac{8}{9} \\
   (3) & \quad -\frac{15}{17} \\
   (4) & \quad \frac{7}{9}
   \end{align*}

5. Which of the following is equal to \( \sin(300^\circ) \)?

   \begin{align*}
   (1) & \quad \sin(60^\circ) \\
   (2) & \quad \sin(30^\circ) \\
   (3) & \quad -\sin(60^\circ) \\
   (4) & \quad -\sin(30^\circ)
   \end{align*}
6. For an angle \( \alpha \) it is known that its reference angle has a sine value of \( \frac{4}{5} \). If the terminal ray of \( \alpha \), when drawn in standard position, falls in the third quadrant then what is the value of \( \cos(\alpha) \)?

(1) \( -\frac{3}{5} \)  
(3) \( -\frac{4}{5} \)  
(2) \( \frac{3}{4} \)  
(4) \( \frac{5}{3} \)

7. The point \( E(-7, -24) \) lies on the circle whose equation is \( x^2 + y^2 = 625 \). If an angle is drawn in standard position and its terminal ray passes through \( E \), what is the value of the sine of this angle?

(1) \(-7\)  
(3) \(-24\)  
(2) \(-\frac{7}{24}\)  
(4) \(-\frac{24}{25}\)

8. If it is known that \( \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \) and \( \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \) then find the value of each of the following. To begin, first determine in which quadrant each angle’s terminal ray lies.

(a) \( \sin\left(\frac{2\pi}{3}\right) \)  
(b) \( \cos\left(\frac{4\pi}{3}\right) \)  
(c) \( \sin\left(\frac{5\pi}{3}\right) \)  
(d) \( \cos\left(-\frac{2\pi}{3}\right) \)

9. Which of the following could not be the value of \( \sin(\theta) \)? Explain how you can tell.

(1) \( -\frac{11}{13} \)  
(3) \( \frac{\sqrt{34}}{5} \)  
(2) \( -\frac{\sqrt{23}}{5} \)  
(4) \( \frac{1}{2} \)

10. A person on a Ferris wheel sits a distance of 45 feet from the Ferris wheel’s center. If they are at an angle of \( 120^\circ \), when measured in standard position, then how high above the center of the wheel are they, to the nearest foot?

(1) 39 feet  
(3) 23 feet  
(2) 12 feet  
(4) 32 feet
BASIC GRAPHS OF SINE AND COSINE
COMMON CORE ALGEBRA II

The sine and cosine functions can be easily graphed by considering their values at the quadrantal angles, those that are integer multiples of $90^\circ$ or $\frac{\pi}{2}$ radians. Due to considerations from physics and calculus, most trigonometric graphing is done with the input angle in units of radians, not degrees.

Exercise #1: Consider the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$, where $x$ is an angle in radians.

(a) By using the unit circle, fill out the table below for selected quadrantal angles.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2\pi$</th>
<th>$-\frac{3\pi}{2}$</th>
<th>$-\pi$</th>
<th>$-\frac{\pi}{2}$</th>
<th>$0$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi$</th>
<th>$\frac{3\pi}{2}$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos(x)$</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Graph both the sine and cosine curves on the grid shown below. Clearly label which curve is which.

(c) The domain and range of the sine and cosine functions are the same. State them below in interval notation.

   Domain: $\quad$ Range:

(d) After how much horizontal distance will both sine and cosine repeat its basic pattern? This is called the **period** of the trigonometric graph. Because these graphs have patterns that repeat they are called **periodic**.
Now we would like to explore the effect of changing the coefficient of the trigonometric function. In essence we would like to look at the graphs of functions of the forms:

\[ y = A \sin(x) \quad \text{and} \quad y = A \cos(x) \]

**Exercise #2:** The grid below shows the graph of \( y = \cos(x) \). Use your graphing calculator to sketch and label each of the following equations. Be sure your calculator is in **Radian Mode**.

\[
\begin{align*}
y &= 3 \cos(x) \\
y &= -4 \cos(x) \\
y &= \frac{3}{2} \cos(x) \\
y &= -2 \cos(x)
\end{align*}
\]

As we can see, this coefficient controls the height that the cosine curves rises and falls above the \( x \)-axis. Its absolute value is given the name **amplitude**. In terms of sound waves it indicates the volume of the sound.

**Exercise #4:** The basic sine function is graphed below. **Without** the use of your calculator, sketch each of the following sine curves on the axes below.

\[
\begin{align*}
y &= 2 \sin(x) \\
y &= 4 \sin(x) \\
y &= -3 \sin(x) \\
y &= -\frac{1}{2} \sin(x)
\end{align*}
\]
BASIC GRAPHS OF SINE AND COSINE

COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. On the grid below, sketch the graphs of each of the following equations based on the basic sine function.

\[ y = \sin(x) \]
\[ y = 3\sin(x) \]
\[ y = -\sin(x) \]
\[ y = -5\sin(x) \]
\[ y = \frac{7}{2}\sin(x) \]

2. On the grid below, sketch the graphs of each of the following equations based on the basic cosine function.

\[ y = \cos(x) \]
\[ y = 4\cos(x) \]
\[ y = -3\cos(x) \]
\[ y = 2.5\cos(x) \]
\[ y = -5.5\cos(x) \]
3. Which of the following represents the range of the trigonometric function \( y = 7 \sin(x) \)?

   (1) \((-7, 7)\)  
   (2) \([-7, 7]\)  
   (3) \([0, 7]\)  
   (4) \((-7, 7)\)

4. Which of the following is the period of \( y = \cos(x) \)?

   (1) \(\pi\)  
   (2) \(2\)  
   (3) \(2\pi\)  
   (4) \(\frac{3\pi}{2}\)

5. Which of the following equations describes the graph shown below?

   (1) \(y = 3\cos(x)\)
   (2) \(y = -3\cos(x)\)
   (3) \(y = 3\sin(x)\)
   (4) \(y = -3\sin(x)\)

6. Which of the following equations represents the periodic curve shown below?

   (1) \(y = 4\cos(x)\)
   (2) \(y = -4\cos(x)\)
   (3) \(y = 4\sin(x)\)
   (4) \(y = -4\sin(x)\)

7. Which of the following lines when drawn would not intersect the graph of \( y = 6\sin(x) \)?

   (1) \(x = 8\)  
   (2) \(x = 3\)  
   (3) \(y = -4\)  
   (4) \(y = 9\)
VERTICAL SHIFTING OF SINUSOIDAL GRAPHS
COMMON CORE ALGEBRA II

Any graph primarily comprised of either the sine or cosine function is known as sinusoidal. These graphs can be stretched vertically, as we saw in the last lesson. Other transformations can occur as well. Today we will explore graphs of equations of the form:

\[ y = A \sin(x) + C \quad \text{and} \quad y = A \cos(x) + C \]

Since we already understand the effect of \( A \) on the graph, it is now time to review the effect of adding a constant to an equation.

**Exercise #1:** Consider the function \( f(x) = \sin(x) + 3 \).

(a) How would the graph of \( y = \sin(x) \) be shifted to produce the graph of \( f(x) \)?

(b) On the grid to the right is the basic sine curve, \( y = \sin(x) \). On the same grid, sketch the graph of \( f(x) \).

**Exercise #2:** Consider the function \( y = 2 \cos(x) + 1 \).

(a) Using your calculator, sketch the graph on the grid to the right.

(b) Give the equation of a horizontal line that this curve rises and falls two units above. Sketch this line on the graph.

(c) State the range of this trigonometric function in interval notation.
For curves that have the general form $y = A \sin(x) + C$ and $y = A \cos(x) + C$ the value $C$ is called the **midline** or **average value** of the trigonometric function. It is the height or horizontal line that the sinusoidal curve rises and falls above and below by a distance of $|A|$ (the amplitude).

**Exercise #3:** Sketch and label the functions $y = 4 \sin(x) - 2$ and $y = -2 \cos(x) + 3$ on the grid below. Try them first without your calculator and then use it to help or verify your graphs. Then, state the ranges of each of the equations in interval notation.

**Range of $y = 4 \sin(x) - 2$:**

**Range of $y = -2 \cos(x) + 3$:**

**Exercise #4:** Determine the range of each of the following trigonometric functions. Express your answer in interval notation.

(a) $y = 7 \sin(x) + 4$  
(b) $y = -5 \cos(x) + 2$  
(c) $y = 25 \sin(x) + 35$

**Exercise #5:** The graph below shows a sinusoidal curve of the form $y = A \sin(x) + C$. Determine the values of $A$ and $C$. Show how you arrived at your results.
FLUENCY

1. Sketch each of the following equations on the graph grid below. Label each with its equation.

\[ y = 4 \sin(x) + 2 \]

\[ y = 2 \cos(x) - 4 \]

\[ y = -\sin(x) + 4 \]

2. Graph and label both of the curves below. Then, state their intersection points (in other words, solve the system of equations shown below).

\[ y = 4 \cos(x) + 1 \]

\[ y = -\cos(x) - 4 \]

Intersection Points:
4. The following graph can be described using an equation of the form \( y = A \cos(x) + C \). Determine the values of \( A \) and \( C \). Show how you arrived at your answers.

![Graph of \( y = A \cos(x) + C \)]

5. The following graph can be described using an equation of the form \( y = A \sin(x) + C \). Determine the values of \( A \) and \( C \). Show how you arrived at your answers.

![Graph of \( y = A \sin(x) + C \)]

6. State the range of each of the following sinusoidal functions in interval form.
   (a) \( y = 10 \sin(x) - 3 \)  
   (b) \( y = -8 \cos(x) + 2 \)  
   (c) \( y = 22 \sin(x) + 30 \)

7. When graphed, the line \( y = 14 \) would not intersect the graph of which of the following functions?
   (1) \( y = 5 \cos(x) + 9 \)  
   (2) \( y = -6 \cos(x) + 10 \)  
   (3) \( y = 2 \sin(x) + 15 \)  
   (4) \( y = 3 \sin(x) + 20 \)

8. Which of the following functions has a maximum value of 25?
   (1) \( y = 25 \sin(x) + 12 \)  
   (2) \( y = -10 \cos(x) + 35 \)  
   (3) \( y = 8 \cos(x) + 17 \)  
   (4) \( y = 5 \sin(x) + 15 \)
THE FREQUENCY AND PERIOD OF A SINUSOIDAL GRAPH
COMMON CORE ALGEBRA II

A final transformation will allow us to horizontally stretch and compress sinusoidal graphs. It is important to be able to do this, especially when modeling real-world phenomena, because most periodic functions do not have a period of \(2\pi\). The first exercise will illustrate the pattern.

**Exercise #1:** On the grid below is a graph of the function \(y = 3\cos(x)\).

(a) Using your calculator, sketch the graph of \(y = 3\cos(2x)\) on the same axes.

(b) How many full cycles or periods of this function now fit within \(2\pi\) radians?

(c) Using your calculator, sketch the graph of \(y = 3\cos\left(\frac{1}{2}x\right)\) on the same axes.

(d) How many full cycles or periods of this function now fit within \(2\pi\) radians?

The period, \(P\), of a sinusoidal function is an extremely important concept. It is defined as the minimum horizontal shift needed for the function to repeat its fundamental pattern. The period for the basic sinusoidal graphs is \(2\pi\). Clearly, from our first exercise, the period of the function depends on the coefficient \(B\) in the general equations \(y = A\sin(Bx)\) and \(y = A\cos(Bx)\). This coefficient, \(B\), is known as the frequency.

**Exercise #2:** Consider the graphs from Exercise #1. For each below, state the frequency and period.

(a) \(y = 3\cos(x)\)  
   Frequency, \(B = \)  
   Period, \(P = \)

(b) \(y = 3\cos(2x)\)  
   Frequency, \(B = \)  
   Period, \(P = \)

(c) \(y = 3\cos\left(\frac{1}{2}x\right)\)  
   Frequency, \(B = \)  
   Period, \(P = \)
Clearly we can see from Exercise #2 that the frequency and period are inversely related, that is as one increases the other decreases and vice versa.

**Exercise #3:** Examine the results from Exercise #2. What is true about the product of the period, \( P \), and the frequency, \( B \)? Write an equation for this relationship.

**Exercise #4:** Determine the period of each of the following sinusoidal functions. Express your answers in exact form.

(a) \( y = 6 \sin(4x) \)  
(b) \( y = 8 \cos \left( \frac{\pi}{3} x \right) \)  
(c) \( y = -12 \sin \left( \frac{2}{3} x \right) \)

**Exercise #5:** Sketch the function \( y = 2 \sin(4x) \) on the grid below for one full period to the left and right of the \( y \)-axis. Label the scale on your axes.

**Exercise #6:** The heights of the tides can be described using a sinusoidal model of the form \( y = A \cos(Bx) + C \). If high tides are separated by 24 hours, which of the following gives the frequency, \( B \), of the curve?

(1) 12  
(2) \( \frac{\pi}{24} \)  
(3) \( \frac{\pi}{12} \)  
(4) \( \frac{\pi}{6} \)
THE FREQUENCY AND PERIOD OF A SINUSOIDAL GRAPH
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. For each of the following sinusoidal functions, determine its period in exact terms of pi.

   (a) \( y = 6 \sin(10x) \)  
   (b) \( y = -2 \cos(8x) \)  
   (c) \( y = 7 \sin\left(\frac{1}{3}x\right) \)

   (d) \( y = \frac{2}{3} \cos\left(\frac{4}{3}x\right) \)  
   (e) \( y = 8 \sin(0.25x) \)  
   (f) \( y = 2.5 \cos(0.4x) \)

2. For each of the following sinusoidal functions, determine its exact period.

   (a) \( y = 5 \sin\left(\frac{2\pi}{7}x\right) \)  
   (b) \( y = 5 \cos\left(\frac{2\pi}{365}t\right) + 12 \)  
   (c) \( y = -8 \sin\left(\frac{\pi}{9}x\right) - 1 \)

3. If the period of a sinusoidal function is equal to 18, which of the following gives its frequency?

   (1) \( \frac{\pi}{9} \)  
   (2) \( 18\pi \)  
   (3) \( \frac{\pi}{18} \)  
   (4) \( 6\pi \)

4. It is known that a particular sine curve repeats its fundamental pattern after every \( \frac{2\pi}{7} \) units along the x-axis. Which of the following is the frequency of this curve?

   (1) \( \frac{2}{7} \)  
   (2) \( 7 \)  
   (3) \( \frac{7}{2} \)  
   (4) \( 14 \)
5. When the period of a sine function doubles, the frequency
   (1) doubles.           (3) is halved.
   (2) increases by 2.    (4) decreases by 2.

6. Which of the following graphs shows the relationship between the frequency, $B$, and the period, $P$, of a sinusoidal graph? Experiment on your calculator. Graph the expression $P = \frac{2\pi}{B}$.

7. Consider the curve whose equation is $y = -2\cos\left(\frac{\pi}{8}x\right) + 3$.
   (a) Determine the exact period of this sinusoidal function.
   (b) What is the amplitude of this sinusoidal function?
   (c) What is the midline value of this sinusoidal function?
   (d) Sketch the function on the axes for a full period on both sides of the $y$-axis. Label the scale on your $x$-axis.
**SINUSOIDAL MODELING**

**COMMON CORE ALGEBRA II**

The sine and cosine functions can be used to model a variety of real-world phenomena that are periodic, that is, they repeat in predictable patterns. The key to constructing or interpreting a sinusoidal model is understanding the physical meanings of the coefficients we’ve explored in the last three lessons.

### SINUSOIDAL MODEL COEFFICIENTS

For \( y = A \sin(Bx) + C \) and \( y = A \cos(Bx) + C \)

- \(|A|\) the **amplitude** or distance the sinusoidal model rises and falls above its midline
- \(C\) the **midline** or average \(y\)-value of the sinusoidal model
- \(B\) the **frequency** of the sinusoidal model – related to the **period**, \(P\), by the equation \(BP = 2\pi\)
- \(P\) the **period** of the sinusoidal model – the minimum distance along the \(x\)-axis for the cycle to repeat

**Exercise #1:** The tides in a particular bay can be modeled with an equation of the form \( d = A \cos(Bt) + C \), where \(t\) represents the number of hours since high-tide and \(d\) represents the depth of water in the bay. The maximum depth of water is 36 feet, the minimum depth is 22 feet and high-tide is hit every 12 hours.

(a) On the axes, sketch a graph of this scenario for two full periods. Label the points on this curve that represent high and low tide.

(b) Determine the values of \(A\), \(B\), and \(C\) in the model. Verify your answers and sketch are correct on your calculator.

(c) Tanker boats cannot be in the bay when the depth of water is less or equal to 25 feet. Set up an inequality and solve it graphically to determine all points in time, \(t\), on the interval \(0 \leq t \leq 24\) when tankers cannot be in the bay. Round all times to the nearest tenth of an hour.
**Exercise #2:** The height of a yo-yo above the ground can be well modeled using the equation \( h = 1.75 \cos(\pi t) + 2.25 \), where \( h \) represents the height of the yo-yo in feet above the ground and \( t \) represents time in seconds since the yo-yo was first dropped from its maximum height.

(a) Determine the maximum and minimum heights that the yo-yo reaches above the ground. Show the calculations that lead to your answers.

(b) How much time does it take for the yo-yo to return to the maximum height for the first time?

**Exercise #3:** A Ferris wheel is constructed such that a person gets on the wheel at its lowest point, five feet above the ground, and reaches its highest point at 130 feet above the ground. The amount of time it takes to complete one full rotation is equal to 8 minutes. A person’s vertical position, \( y \), can be modeled as a function of time in minutes since they boarded, \( t \), by the equation \( y = A \cos(Bt) + C \). Sketch a graph of a person’s vertical position for one cycle and then determine the values of \( A \), \( B \), and \( C \). Show the work needed to arrive at your answers.

**Exercise #4:** The possible hours of daylight in a given day is a function of the day of the year. In Poughkeepsie, New York, the minimum hours of daylight (occurring on the Winter solstice) is equal to 9 hours and the maximum hours of daylight (occurring on the Summer solstice) is equal to 15 hours. If the hours of daylight can be modeled using a sinusoidal equation, what is the equation’s amplitude?

- (1) 6
- (2) 12
- (3) 3
- (4) 4
APPLICATIONS

1. A ball is attached to a spring, which is stretched and then let go. The height of the ball is given by the sinusoidal equation \( y = -3.5 \cos \left( \frac{4\pi}{5} t \right) + 5 \), where \( y \) is the height above the ground in feet and \( t \) is the number of seconds since the ball was released.

(a) At what height was the ball released at? Show the calculation that leads to your answer.

(b) What is the maximum height the ball reaches?

(c) How many seconds does it take the ball to return to its original position?

(d) Draw a rough sketch of one complete period of this curve below. Label maximum and minimum points.

2. An athlete was having her blood pressure monitored during a workout. Doctors found that her maximum blood pressure, known as systolic, was 110 and her minimum blood pressure, known as diastolic, was 70. If each heartbeat cycle takes 0.75 seconds, then determine a sinusoidal model, in the form \( y = A \sin (Bt) + C \), for her blood pressure as a function of time \( t \) in seconds. Show the calculations that lead to your answer.
3. On a standard summer day in upstate New York, the temperature outside can be modeled using the sinusoidal equation \( O(t) = 11 \cos \left( \frac{\pi}{12} t \right) + 71 \), where \( t \) represents the number of hours since the peak temperature for the day.

(a) Sketch a graph of this function on the axes below for one day.

---

(b) For \( 0 \leq t \leq 24 \), graphically determine all points in time when the outside temperature is equal to 75 degrees. Round your answers to the nearest tenth of an hour.

---

4. The percentage of the moon’s surface that is visible to a person standing on the Earth varies with the time since the moon was full. The moon passes through a full cycle in 28 days, from full moon to full moon. The maximum percentage of the moon’s surface that is visible is 50%. Determine an equation, in the form \( P = A \cos(Bt) + C \) for the percentage of the surface that is visible, \( P \), as a function of the number of days, \( t \), since the moon was full. Show the work that leads to the values of \( A \), \( B \), and \( C \).

5. Evie is on a swing thinking about trigonometry (no seriously!). She realizes that her height above the ground is a periodic function of time that can be modeled using \( h = 3 \cos \left( \frac{\pi}{2} t \right) + 5 \), where \( t \) represents time in seconds. Which of the following is the range of Evie’s heights?

   (1) \( 2 \leq h \leq 8 \)       (3) \( 3 \leq h \leq 5 \)
   (2) \( 4 \leq h \leq 8 \)       (4) \( 2 \leq h \leq 5 \)
The two most important circular functions are sine and cosine. But, recall from Common Core Geometry, that there was a third trigonometric function known as tangent. There are actually three more that you will learn about in the next lesson, but in this lesson we will only look at the tangent function. Let’s recall how it was defined using right triangle trigonometry.

**Exercise #1:** For each of the right triangles below, state the values of \( \sin(\theta) \), \( \cos(\theta) \), and \( \tan(\theta) \).

(a)  

(b)  

Notice that we can have missing sides of a right triangle and still use the Pythagorean Theorem to find these ratios (many of which may involve radicals and thus irrational numbers). Just as we did with sine and cosine, we can also define tangent in terms of the unit circle.

**Exercise #2:** Consider the unit circle shown below. Answer the following questions based on this diagram.

(a) If \((x, y)\) represents a point on the circle, how can we define the tangent function in terms of this point?

(b) Given how we define sine and cosine, how can we define the tangent function in terms of sine and cosine?
### Exercise #3: Use the unit circle and the definition of tangent in terms of sine and cosine to find the value for each of the following. Do not leave any complex fractions.

(a) $\tan(60^\circ)$  
(b) $\tan(45^\circ)$  
(c) $\tan(150^\circ)$  
(d) $\tan(180^\circ)$

Because the tangent function involves division, there is a chance it could be undefined.

**Exercise #4:** Consider the angle $\theta = 90^\circ$ or $\frac{\pi}{2}$ radians.

(a) State the values of sine and cosine at this angle.  
(b) Why would the value of $\tan(90^\circ)$ be undefined?

**Exercise #5:** At which of the following angles is tangent undefined?

1. $\theta = 0^\circ$  
2. $\theta = 270^\circ$  
3. $\theta = 120^\circ$  
4. $\theta = -180^\circ$

On a final note, it is interesting that if we know sine or cosine of an angle and the quadrant of the angle we can find the other two missing trigonometric values.

**Exercise #6:** Determine the value of $\cos(\theta)$ and $\tan(\theta)$ if $\sin(\theta) = \frac{5}{13}$ and the terminal ray of $\theta$ lies in the second quadrant.

**Exercise #7:** If $\cos(\theta) = a$, where $a > 0$, and the terminal ray of $\theta$ lies in the fourth quadrant, then which of the following gives the value of $\tan(\theta)$ in terms of $a$.

1. $\frac{1-a}{a}$  
2. $-\sqrt{1-a^2}$  
3. $\sqrt{1-a^2}$  
4. $\frac{-\sqrt{1-a^2}}{a}$
THE TANGENT FUNCTION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Using the unit circle diagram, find the exact values for each of the following. Don’t leave any complex fractions. Show how you arrived at your final answers. You can check using your calculator, but decimal answers should not be given. If the value of tangent is undefined, state UND.

(a) \( \tan(60^\circ) \)
(b) \( \tan(150^\circ) \)
(c) \( \tan(225^\circ) \)

(d) \( \tan(270^\circ) \)
(e) \( \tan\left(\frac{2\pi}{3}\right) \)
(f) \( \tan\left(\frac{11\pi}{6}\right) \)

2. The point \((0.28, 0.96)\) lies on the unit circle. Which of the following is closest to the tangent of an angle drawn in standard position whose terminal ray passes through this point?

(1) 3.43  
(2) 1.73  
(3) 0.29  
(4) 0.42

3. At which of the following angles is the tangent function undefined?

(1) \( \theta = 180^\circ \)  
(2) \( \theta = -90^\circ \)  
(3) \( \theta = 45^\circ \)  
(4) \( \theta = -360^\circ \)

4. Which of the following values of \( x \) is not in the domain of \( g(x) = \tan(2x) \)? Hint – you will be multiplying each of these values by 2 before finding its tangent.

(1) \( 45^\circ \)  
(2) \( 0^\circ \)  
(3) \( 180^\circ \)  
(4) \( 90^\circ \)
5. Determine whether each function in the tables below is positive, (+), or negative, (−), for angles whose terminal rays lie in the respective quadrants. Use values of sine and cosine to determine the sign of the tangent function.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>cos(θ)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sin(θ)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tan(θ)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. For an angle α it is known that tan(α) > 0 and sin(α) < 0. The terminal ray of α when drawn in standard position must lie in which quadrant? Hint: see the table in #5 for help.

(1) I   (3) III
(2) II  (4) IV

7. For each of the following problems, the value of either sine or cosine of an angle is given along with the quadrant in which the terminal ray of the angle lies. For each, produce the values of the two missing trigonometric functions. Some of your answers will have radicals (irrational numbers) in them. You should not leave complex fractions.

(a) cos(α) = $\frac{-3}{5}$ and $A$ terminates in quadrant II.
(b) sin(α) = $\frac{1}{3}$ and $α$ terminates in quadrant I.
(c) sin(θ) = $\frac{-5}{13}$ and $θ$ terminates in quadrant III.
(d) cos(B) = $\frac{2}{5}$ and $B$ terminates in quadrant IV.
THE RECIPROCAL TRIG FUNCTIONS
COMMON CORE ALGEBRA II

We have now seen three primary trigonometric functions, the sine, cosine, and tangent functions. Each of these can be defined in terms of either ratios of the sides of a right triangle or the unit circle. For each of these functions, though, there exists what is known as a reciprocal function. Their definitions are shown below.

<table>
<thead>
<tr>
<th>The Other Four Trigonometric Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Secant: ( \sec(\theta) = \frac{1}{\cos(\theta)} )</td>
</tr>
<tr>
<td>2. Cosecant: ( \csc(\theta) = \frac{1}{\sin(\theta)} )</td>
</tr>
<tr>
<td>3. Cotangent: ( \cot(\theta) = \frac{1}{\tan(\theta)} ) or equivalently ( \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} )</td>
</tr>
</tbody>
</table>

**Exercise #1:** Considering your work with sine and cosine, evaluate each of the following. Express your answers in exact and simplest form.

(a) \( \sec(60^\circ) \)  
(b) \( \cot(150^\circ) \)  
(c) \( \csc\left(\frac{3\pi}{4}\right) \)

**Exercise #2:** Which of the following is closest to the value of \( \sec(52^\circ) \)?

(1) 0.62  
(2) 1.62  
(3) 0.36  
(4) 2.48  

Because each of these reciprocal trigonometric functions has a variable denominator, there will be angles at which these denominators are zero and hence the function is undefined.

**Exercise #3:** Which of the following values of \( x \) is *not* in the domain of \( y = \csc(x) \)?

(1) \( x = 180^\circ \)  
(2) \( x = 60^\circ \)  
(3) \( x = 90^\circ \)  
(4) \( x = 135^\circ \)
Because each of these functions is dependent on sine and/or cosine, it is possible to determine the sign (positive or negative nature) of each based on the quadrant of the input angle.

**Exercise #4:** Determine the sign of each of the following trigonometric functions in the quadrant specified.

(a) \( \cot(\beta) \) for \( \beta \) in quad. II  
(b) \( \sec(\beta) \) for \( \beta \) in quad. IV  
(c) \( \csc(\beta) \) for \( \beta \) in quad. III

**Exercise #5:** If \( \cot(\theta) < 0 \) and \( \sec(\theta) > 0 \) then \( \theta \) could be which of the following angles?

1. \( \theta = 48^\circ \)  
2. \( \theta = 310^\circ \)  
3. \( \theta = 122^\circ \)  
4. \( \theta = 225^\circ \)

We should also be able to produce all of the trigonometric ratios (all SIX of them) if we are given a right triangle.

**Exercise #6:** A right triangle is shown below with sides of length \( a \) and \( b \).

(a) Find the length of the hypotenuse in terms of \( a \) and \( b \). Label on the diagram.

(b) State the value of each of the following trigonometric ratios in terms of the constants \( a \) and \( b \).

\[
\sin A = \quad \csc A = \\
\cos A = \quad \sec A = \\
\tan A = \quad \cot A =
\]

**Exercise #7:** If \( \alpha \) is an angle whose terminal ray lies in the fourth quadrant and \( \cos \alpha = \frac{1}{3} \), then determine the exact value of \( \csc \alpha \). Show how you arrived at your answer.
THE RECIPROCAL TRIG FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Determine the value of each of the following in exact and simplest form (leave no complex fractions).
   (a) \( \csc(30^\circ) \)  
   (b) \( \cot(90^\circ) \)  
   (c) \( \sec(180^\circ) \)  
   (d) \( \cot \left( \frac{\pi}{3} \right) \)  
   (e) \( \csc \left( \frac{3\pi}{2} \right) \)  
   (f) \( \sec \left( \frac{5\pi}{4} \right) \)

2. Use your calculator to determine the value of each of the following to the nearest hundredth.
   (a) \( \cot(115^\circ) \)  
   (b) \( \sec(312^\circ) \)  
   (c) \( \csc(245^\circ) \)

3. In simplest radical form, \( \sec(135^\circ) \) is equal to
   (1) \( -\frac{\sqrt{2}}{3} \)  
   (3) \( -\frac{\sqrt{2}}{2} \)  
   (2) \( -\sqrt{2} \)  
   (4) \( -\frac{\sqrt{3}}{2} \)

4. Which of the following is nearest to the value of \( \cot(220^\circ) \)?
   (1) 1.19  
   (3) −2.74  
   (2) 3.17  
   (4) −0.85
5. For which of the following values of \( \alpha \) is \( \cot(\alpha) \) undefined?

(1) 60\(^\circ\)  (3) 180\(^\circ\)
(2) 90\(^\circ\)  (4) 135\(^\circ\)

6. For which angle, \( \beta \), below will \( \sec(\beta) \) not exist?

(1) 30\(^\circ\)  (3) 180\(^\circ\)
(2) 45\(^\circ\)  (4) 90\(^\circ\)

7. Determine whether each function in the tables below is positive, (+), or negative, (−), for angles whose terminal rays lie in the respective quadrants. Use the table in part (a) to help create the table in (b).

(a)   (b)
| cos(\( \theta \)) | I  | II | III | IV |
| sin(\( \theta \)) |    |    |     |    |

(b)   
| tan(\( \theta \)) | I  | II | III | IV |
| cot(\( \theta \))  |    |    |     |    |
| sec(\( \theta \))  |    |    |     |    |
| csc(\( \theta \))  |    |    |     |    |

8. For the angle \( \beta \) it is known that \( \csc(\beta) > 0 \) and \( \sec(\beta) < 0 \). When drawn in standard position, the terminal ray of \( \beta \) lies in quadrant

(1) I  (3) III
(2) II  (4) IV

9. The angle \( \theta \) when drawn in standard position has its terminal ray in the second quadrant. If it is known that \( \sin \theta = \frac{5}{13} \) then determine the values of all of the remaining trigonometric functions.

(a) cos \( \theta \)  (b) tan \( \theta \)  (c) sec \( \theta \)
(d) csc \( \theta \)  (e) cot \( \theta \)
UNIT #12

PROBABILITY

Lesson #1 – Introduction to Probability
Lesson #2 – Sets and Probability
Lesson #3 – Adding Probabilities
Lesson #4 – Conditional Probability
Lesson #5 – Independent and Dependent Events
Lesson #6 – Multiplying Probabilities
INTRODUCTION TO PROBABILITY
COMMON CORE ALGEBRA II

Mathematics seeks to quantify and model just about everything. One of the greatest challenges is to try to quantify chance. But that is exactly what probability seeks to do. With probability, we attempt to assign a number to how likely an event is to occur. Terminology in probability is important, so we introduce some basic terms here:

**BASIC PROBABILITY TERMINOLOGY**

1. **Experiment:** Some process that occurs with well defined outcomes.
2. **Outcome:** A result from a single trial of the experiment.
3. **Event:** A collection of one or more outcomes.
4. **Sample Space:** A collection of all of the outcomes of an experiment.

**Exercise #1:** An experiment is run whereby a spinner is spun around a circle with 5 equal sectors that have been marked off as shown.

(a) What is the experiment?

(b) Give one outcome of the experiment.

(c) What is the probability of spinning the spinner and landing on an odd number? What is the event here? What outcomes fall into the event?

The answer from (c) helps us to define the basic formula that dictates all probability calculations:

**THE BASIC DEFINITION OF PROBABILITY**

The probability of an event \( E \) occurring is given by the ratio: \( P(E) = \frac{n(E)}{n(S)} \), where:

- \( n(E) \) is the number of outcomes that fall into the event \( E \)
- \( n(S) \) is the number of outcomes that fall into the sample space

**Exercise #2:** Given the above definition, between what two numbers must ALL probabilities lie? Explain.
When we deal with **theoretical probability** we don’t actually have to run the experiment to determine the probability of an event. We simply have to know the number of outcomes in the sample space and the number of outcomes that fall into our event. Let’s take a look at a slightly more challenging scenario.

**Exercise #3:** A fair coin is flipped three times and the result is noted each time. The sample space consists of **ordered triples** such as \((H, H, T)\), which would represent a head on the first toss, a head on the second toss, and a tail on the third toss.

(a) Draw a **tree diagram** to show all of the different outcomes in the sample space.

(b) List all of the outcomes as ordered triples. How many of them are there?

(c) Find each of the following probabilities based on your answers from (a) and (b):

(i) \(P(\text{all heads})\)

(ii) \(P(\text{exactly 2 heads})\)

(iii) \(P(\text{all heads or all tails})\)

Sometimes we have to quantify chance by using observations that have been made in the real-world. In this case we talk about **empirical probability**. The fundamental equation for probability still stands.

**Exercise #4:** A survey was done by a marketing company to determine which of three sodas was preferred by people in a blind taste test. The results are shown below.

<table>
<thead>
<tr>
<th>Soda</th>
<th>Number who Preferred</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>18</td>
</tr>
<tr>
<td>B</td>
<td>24</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>53</td>
</tr>
</tbody>
</table>

(a) Find the empirical probability that a person selected at random from this group would prefer soda B. Express your answer as a fraction and as a decimal accurate to two decimal places (the standard).

(b) Find the empirical probability that a person selected at random from this group would *not* prefer soda A. Again, express your answer as a fraction and as a decimal accurate to two decimal places.
INTRODUCTION TO PROBABILITY
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following could not be the value of a probability? Explain your choice.

   (1) 53%  \hspace{1cm} (3) \frac{5}{4}

   (2) 0.78  \hspace{1cm} (4) \frac{3}{4}

2. If a month is picked at random, which of the following represent the probability its name will begin with the letter J?

   (1) 0.08  \hspace{1cm} (3) 0.12

   (2) 0.25  \hspace{1cm} (4) 0.33

3. If a coin is tossed twice, which of the following gives the probability that it will land both times heads up or both times tails up?

   (1) 0.75  \hspace{1cm} (3) 0.25

   (2) 0.67  \hspace{1cm} (4) 0.50

4. A spinner is now created with four equal sized sectors as shown. An experiment is run where the spinner is spun twice and the outcome is recorded each time.

   (a) Create a sample space list of ordered pairs that represent the outcomes, such as \((4, 2)\), which represent spinning a 4 on the first spin and a 2 on the second spin.

   (b) Using your answer from (a), determine the probability of obtaining two numbers with a sum of 4.
APPLICATIONS

5. Samuel pulls two coins out of his pocket randomly without replacement. If his pocket contains one nickel, one dime, and one quarter, what is the probability that he pulled more than 20 cents out of his pocket? Justify your work by creating a tree diagram or a sample space.

6. Janice, Tom, John, and Tamara are trying to decide on who will make dinner and who will wash the dishes afterwards. They randomly pull two names out of a hat to decide, where the first name drawn will make dinner and the second will do the dishes. Determine the probability that the two people pulled will have first names beginning with the same letter. Assume the same person cannot be picked for both.

7. A blood collection agency tests 50 blood samples to see what type they are. Their results are shown in the table below.

(a) If a blood sample is picked at random, what is the probability it will be type B?

(b) If a blood sample is picked at random, what is the probability it will not be type O?

(c) Are the two probabilities you calculated in (a) and (b) theoretical or empirical? Explain your choice.
SETS AND PROBABILITY
COMMON CORE ALGEBRA II

Since the basic calculation within probability involves counting the number of outcomes that fit into a particular event, it makes sense to have a tool to visualize and keep track of all of the outcomes in a sample space. We will do this by using sets. Recall the basic definition of a set:

**Set Definition**
A set is simply a collection of things (numbers, objects, etcetera) that satisfy a well-defined criteria. The things that are contained in the set are called the elements of the set.

**Exercise #1:** The set A is defined as the collection of all integers that are greater than 0 and less than 10.
(a) Write out set A in roster form.
(b) Show set A in Venn Diagram form. This will be a very simple Venn Diagram.

(c) A subset is any set whose elements are all contained within another set. Give two possible rules that could define subsets of A and then write the sets as B and C in roster form. Do sets B and C have any elements in common?

Set B’s Definition: ____________________  \[ B = \]
Set C’s Definition: ____________________  \[ C = \]

Let’s get back to a bit of probability.

**Exercise #2:** Consider an experiment where we first toss a coin and note the outcome and then roll a six-sided die and note the outcome.
(a) Write a set of ordered pairs, such as \((H, 4)\), that represents all outcomes for this experiment. Recall that this is called the sample space. We will generally call this set S.
(b) Write a set of ordered pairs that represents the event of getting a tail and an even number. Call this set A.

(c) The complement of a set A will be all of the events in the sample space S that do not fall into set A. Write out the complement of set A. We’ll call this set B.
(d) Find \(P(A)\) and \(P(B)\).
A set and its complement are important when we look at probability because all outcomes either fall into an event or into its complement, but not both. Different textbooks use different notations to denote complements. Since the notation is not universal, we will simply refer to complements by name instead of by symbol.

**Exercise #3:** Consider rolling a single six-sided die and recording the result. Let set A be the event of rolling a number greater than 4 and let set B be the complement of set A.

(a) Draw a Venn Diagram that illustrates the sample space, S, and sets A and B.

(b) Find \( P(A) \) and \( P(B) \).

(c) What is true of the sum \( P(A) + P(B) \)?
(d) Prove that the sum of the probability of an event with the probability of its complement will always be 1.

We use the relationship developed in (d) all the time without even thinking about it. Try the following.

**Exercise #4:** Answer each of the following problems by using the relationship developed in Exercise #3(d).

(a) If the probability I will draw a red marble from a bag is \( \frac{3}{17} \), what is the probability that I won’t draw a red marble from a bag?

(b) If the probability that it will rain tomorrow is 20%, what is the probability that it won’t rain tomorrow?

In theoretical probability calculations, the sets that make up the sample spaces can get difficult to write out. It is good to remember things like tree diagrams to help.

**Exercise #5:** Two four-sided die are rolled and the number on each is noted.

(a) Draw a tree diagram that represents all outcomes in the sample space. How many are there?

(b) What is the probability that you don’t get two of the same number?
APPLICATION

1. Consider the experiment of picking one of the 12 months at random.
   (a) Write down that sample space, S, for this experiment. What is the value of \( n(S) \)?
   (b) Let E be the event (set) of picking a month that begins with the letter J. Write out the elements of E.

   (c) What is the probability of E, i.e. \( P(E) \)?
   (d) What is the probability of picking a month that does not start with the letter J?

2. Consider the set, A, of all integers from 1 to 10 inclusive (that means the 1 and the 10 are included in this set). Give a set B that is a subset of A. State its definition and list its elements in roster form. Then give a set C that is the complement of B.

   Set B’s Definition: _______________________________________  Set B:

   Set C:

3. If A and B are complements, then which of the following is true about the probability of B based on the probability of A?

   \[ P(B) = P(A) + 1 \quad \text{(3) } P(B) = \frac{1}{P(A)} \]
   \[ P(B) = 1 - P(A) \quad \text{(4) } P(B) = P(A) - 1 \]

4. If a fair coin is flipped three times, the probability it will land heads up all three times is \( \frac{1}{8} \). Which of the following is the probability that when a coin is flipped three times at least one tail will show up?

   \[ \frac{7}{8} \quad \text{(3) } \frac{3}{2} \]
   \[ \frac{1}{8} \quad \text{(4) } \frac{1}{2} \]
5. A four-sided die, in the shape of a tetrahedron, is rolled twice and the number rolled is recorded each time.

(a) Draw a tree-diagram that shows the sample space, $S$, of this experiment. How many elements are in $S$?

(b) Let $E$ be the event of rolling two numbers that have an odd product. List all of the elements of $E$ as ordered pairs.

(c) What is the probability that the two rolled numbers have a product that is odd?

(d) What is the probability that the two rolled numbers have a product that is even?

**REASONING**

6. Consider the set of all integers from 1 to 10, i.e. \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, to be our sample space, $S$.

Let $A$ be the set of all integers in $S$ that are even and let $B$ be the set of all integers in $S$ that are multiples of 3.

Fill in the circles of the Venn diagram with elements from $S$. If an element lies in both sets, place it in the overlapping region.

7. Find in the following:

   $n(A) =$  
   $n(B) =$

8. Why is the following equation not true? Explain.

   $n(S) = n(A) + n(B)$
There are times that we want to determine the probability that either event A happened or event B happened. To do this, we need to be able to account for all of the outcomes that fall into either one of the two events. Let's see how this looks given a simple Venn diagram.

**Exercise #1:** Consider the spinner shown below that has been divided into eight equally sized sectors of a circle. The spinner is spun once. In this experiment we will let A be the event of it landing on an even and B be the event of it landing on a prime number.

Fill in the Venn Diagram below with the actual numbers from the spinner.

When we have two (or more) sets, we can talk about their union and their intersection. Their technical definitions are given below.

**THE UNION AND INTERSECTION OF TWO SETS**

For two sets, A and B, their union, OR, and their intersection, AND, are given by:

1. **Union:**
   \[ A \cup B = \{ x : x \text{ is in } A \text{ or } x \text{ is in } B \} \]

2. **Intersection:**
   \[ A \cap B = \{ x : x \text{ is in } A \text{ and } x \text{ is in } B \} \]

**Exercise #2:** From Exercise #1 write out the following two sets:

(a) A or B (The Union):

(b) A and B (The Intersection):

**Exercise #3:** From Exercise #2, why is the equation \( n(A \cup B) = n(A) + n(B) \) generally not true? What would be the correct modification to make it true? Use the last example to help explain.
Two-way frequency charts give us a great example of how events or sets can combine (union) and overlap (intersection). Let's take a look at this and develop some ideas about probability along the way.

**Exercise #4:** A small high school surveyed 52 of its seniors about their plans after they graduate. They found the following data and wanted to analyze it based on gender. In this case, if we pick a student at random we can place them into one of four events:

\[
\begin{align*}
M &= \text{Male} \\
F &= \text{Female} \\
C &= \text{Going to College} \\
N &= \text{Not going to college}
\end{align*}
\]

(a) Give the values for each of the following:

(i) \( n(M) = \) 

(ii) \( n(F) = \) 

(iii) \( n(C) = \) 

(iv) \( n(N) = \) 

(v) \( n(M \text{ and } C) = \) 

(vi) \( n(F \text{ and } C) = \) 

(vii) \( n(F \text{ or } C) = \) 

(b) What is the probability that a person picked at random would be a female who is going to college? Represent this using either a union or an intersection.

(c) What is the probability that a person picked at random would be a female or someone going to college? Represent this using either a union or an intersection.

(d) Explain why \( P(F \text{ or } C) \neq P(F) + P(C) \)?

(e) Fill in the general probability law based on (d):

\[
P(A \text{ or } B) =
\]

Sometimes we can avoid the probability law that we encounter in (e) by simply keeping careful track of what elements of the sample space are in both of our sets and making sure we don't count any element twice.

**Exercise #5:** A standard six-sided die is rolled once. Find the probability that the number rolled was either an even or a multiple of three. Represent this problem and the sets involved using a Venn diagram. Even though you don't need it, verify the probability addition rule from Exercise #4 (e).

There are some situations, though, where the probability addition rule is unavoidable.

**Exercise #6:** Insurance companies typically try to sell many different policies to the same customers. At one such company, 56% of all of the customers have car insurance policies, 48% have home insurance policies, and 18% have both. A customer is picked at random.

(a) Find the probability that she or he has at least one of the policies.

(b) Find the probability that she or he has neither of the policies.
**FLUENCY**

1. Given the two sets below, give the sets that represent their union and their intersection.

   \[ A = \{3, 5, 7, 9, 11, 13\} \quad B = \{1, 5, 9, 13, 17\} \]

   (a) Union: A or B =
   
   (b) Intersection: A and B =

2. Using sets A and B from #1, verify the addition law for the union of two sets:

   \[ n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B) \]

**APPLICATIONS**

3. Red Hook High School has 480 freshmen. Of those freshmen, 333 take Algebra, 306 take Biology, and 188 take both Algebra and Biology. Which of the following represents the number of freshmen who take at least one of these two classes?

   (1) 639 (3) 451
   
   (2) 384 (4) 425

4. Evie was doing a science fair project by surveying her biology class. She found that of the 30 students in the class, 15 had brown hair and 17 had blue eyes and 6 had neither brown hair nor blue eyes. Determine the number of students who had brown hair and blue eyes. Use the Venn Diagram below to help sort the students if needed.
5. A standard six-sided die is rolled and its outcome noted. Which of the following is the probability that the outcome was less than three or even?

(1) \(\frac{2}{3}\)  \hspace{1cm} (3) \(\frac{5}{6}\)

(2) \(\frac{1}{3}\)  \hspace{1cm} (4) \(\frac{1}{6}\)

6. Historically, a given day at the beginning of March in upstate New York has a 18% chance of snow and a 12% chance of rain. If there is a 4% chance it will rain and snow on a day, then which of the following represents the probability that a day in early March would have either rain or snow?

(1) 0.30  \hspace{1cm} (3) 0.02

(2) 0.34  \hspace{1cm} (4) 0.26

7. A survey was done of students in a high school to see if there was a connection between a student's hair color and her or his eye color. If a student is chosen at random, find the probability of each of the following events.

(a) The student had black hair.

(b) The student had blue eyes.

(c) The student had brown eyes and black hair.

(d) The student had blue eyes or blond hair.  \hspace{1cm} (e) The student had black hair or blue eyes.

8. A recent survey of the Arlington High School 11th grade students found that 56% were female and 58% liked math as their favorite subject (of course). If 76% of all students are either female or liked math as their favorite subject, then what percent of the 11th graders were female students who liked math as their favorite subject? Show how you arrived at your answer.
When the probability of one event occurring changes depending on other events occurring then we say that there is a **conditional probability**. The language and symbolism of conditional probability can be a bit confusing, but the idea is fairly straightforward and can be developed with two-way frequency charts.

**Exercise #1:** Let's revisit a two-way frequency chart we saw in the last lesson. In this study, 52 graduating seniors were surveyed as to their post-graduation plans and then the results were sorted by gender.

Let the following letters stand for the following events.

- **M** = Male
- **F** = Female
- **C** = Going to College
- **N** = Not going to college

If a person was picked at random, find the probability that the person was

(a) a female, i.e. \( P(F) \)

(b) going to college \( P(C) \)

(c) going to college **given** they are female, i.e. \( P(C \mid F) \). Draw a Venn diagram below to help justify the ratio that you give as the probability.

(d) Which is more likely, that a person picked at random will be going to college, given they are a male, i.e. \( P(C \mid M) \), or that a person will be male, given they are going to college, i.e. \( P(M \mid C) \). Show that calculations for both.

\[
P(C \mid M) \quad P(M \mid C)
\]
We can generalize this process to calculate these conditional probabilities based on a counts and a way to
calculate these probabilities based on other probabilities.

**Exercise #2:** In the generic Venn diagram shown to the right. Each dot represents an equally likely outcome of the sample space. Some of these fall only into event A, some only into event B, some in both events and
some in neither.

(a) Consider the probability of A occurring given that B has occurred. Give a formula for this probability based on counting the number of elements in each set and their intersection.

\[ P(A \mid B) = \]

(b) Divide both of the numerator and denominator in (a) by the number of total elements in the sample space. Then rewrite the formula in (a) in terms of probabilities instead of counts.

\[ P(A \mid B) = \]

It's great when we can count elements that lie in events and their intersection, but sometimes we cannot. For example, let's revisit a relative frequency table that we saw in a previous homework.

**Exercise #3:** A survey was taken to examine the relationship between hair color and eye color. The chart below shows the proportion of the people surveyed who fell into each category. If a person was picked at random, find each of the following conditional probabilities. Show the calculation you used.

(a) Find the probability the person picked had brown eyes given they had blond hair.

\[ P(\text{brown eyes} \mid \text{blond hair}) \]

(b) Find the probability the person had red hair given they had green eyes.

\[ P(\text{red hair} \mid \text{green eyes}) \]

(c) Does having red hair seem have some dependence on having green eyes? How can you tell or quantify this dependence?
**Conditional Probability**

**Common Core Algebra II Homework**

**Fluency**

1. Given that \( P(B \mid A) \) means the probability of event B occurring given that event A will occur or has occurred, which of the following correctly calculates this probability?

   (1) \( \frac{P(B)}{P(A)} \)  
   (2) \( \frac{P(A \text{ and } B)}{P(B)} \)  
   (3) \( \frac{P(A)}{P(B)} \)  
   (4) \( \frac{P(A \text{ and } B)}{P(A)} \)

**Applications**

2. Of the 650 juniors at Arlington High School, 468 are enrolled in Algebra II, 292 are enrolled in Physics, and 180 are taking both courses at the same time. If one of the 650 juniors was picked at random, what is the probability they are taking Physics, if we know they are in Algebra II?

   (1) 0.38  
   (2) 0.62  
   (3) 0.45  
   (4) 0.58

3. Historically, a given day at the beginning of March in upstate New York has a 18% chance of snow and a 12% chance of rain. If there is a 4% chance it will rain and snow on a day, then calculate each of the following:

   (a) the probability it will rain given that it is snowing, i.e. \( P(\text{rain} \mid \text{snow}) \)
   (b) the probability it will snow given that it is raining, i.e. \( P(\text{snow} \mid \text{rain}) \)

4. A spinner is spun around a circle that is divided up into eight equally sized sectors. Find:

   (a) \( P(\text{perfect square} \mid \text{even}) \)  
   (b) \( P(\text{odd} \mid \text{prime}) \)

   (c) What is more likely: getting a multiple of four given we spun an even or getting an odd, given we spun a number greater than 2? Support your answer.
5. A survey was done of commuters in three major cities about how they primarily got to work. The results are shown in the frequency table below. Answer the following conditional probability questions.

(a) What is the probability that a person picked at random would take a train to work given that they live in Los Angeles.

\[
P(\text{train} \mid \text{LA})
\]

(b) What is the probability that a person picked at random would live in New York given that they drive a car to work.

\[
P(\text{NYC} \mid \text{Car})
\]

(c) Is it more likely that a person who takes a train to work lives in Chicago or more likely that a person who lives in Chicago will take a train to work. Support your work using conditional probabilities.

### Frequency Table

<table>
<thead>
<tr>
<th></th>
<th>Car</th>
<th>Train</th>
<th>Walk</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>New York</td>
<td>.05</td>
<td>.25</td>
<td>.10</td>
<td>.40</td>
</tr>
<tr>
<td>Los Angeles</td>
<td>.18</td>
<td>.12</td>
<td>.05</td>
<td>.35</td>
</tr>
<tr>
<td>Chicago</td>
<td>.08</td>
<td>.14</td>
<td>.03</td>
<td>.25</td>
</tr>
<tr>
<td>Total</td>
<td>.31</td>
<td>.51</td>
<td>.18</td>
<td>1.00</td>
</tr>
</tbody>
</table>

### REASONING

6. The formula for conditional probability is: \[ P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \]. Solve this formula for \( P(A \text{ and } B) \).

7. We say two events, \( A \) and \( B \), are independent if the following is true:

\[
P(B \mid A) = P(B) \quad \text{and likewise} \quad P(A \mid B) = P(A)
\]

Interpret what the definition of independent events means in your own words.
INDEPENDENT AND DEPENDENT EVENTS
COMMON CORE ALGEBRA II

In the previous lesson's homework we say how the occurrence of one event could change the probability of another event. When this happens, we say the two events are dependent on one another. When the occurrence of one event has no effect on the probability of another event happening, we say the events are independent.

Exercise #1: Classify each of the following scenarios as having events that are dependent or events that are independent.
(a) A person pulls a red marble out of a bag that has 5 blue and 7 red marbles and does not replace it. Then a person pulls another red marble. Is the probability of pulling the second red marble out dependent on pulling the first red marble? Explain.
(b) A person flips a coin and notes that it comes up heads. Then the person rolls a standard six-sided die and notes that it comes up as a number less than three. Is the probability that the number came up less than three dependent on getting a head when flipping the coin? Explain.

The idea of independence is one that comes fairly naturally, but is important in order to see if there are associations amongst two events. Let's develop a tool to test dependence.

Exercise #2: The spinner below is spun once and its outcome is noted. Let E be the event of getting an even, let P be the event of getting a prime, and let L be the event of getting a number less than 5. Find the following probabilities:
(a) The probability of getting an even, i.e. $P(E)$.
(b) The probability of getting an even given that the outcome was a prime number, i.e. $P(E | P)$
(c) The probability of getting an even given that the outcome was a number less than 5, i.e. $P(E | L)$.
(d) Which event does E depend on, P or L? How can you tell? What is a reasonable test?
**Exercise #3:** A survey of 57 sixth graders was done to determine which subject was their favorite. The results are shown in the table below sorted by gender.

<table>
<thead>
<tr>
<th></th>
<th>Math</th>
<th>English</th>
<th>Social Studies</th>
<th>Science</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Female</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>6</td>
<td>30</td>
</tr>
<tr>
<td>Male</td>
<td>10</td>
<td>4</td>
<td>9</td>
<td>4</td>
<td>27</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
<td>10</td>
<td>19</td>
<td>9</td>
<td>57</td>
</tr>
</tbody>
</table>

(a) Does it appear, based on the data in this table, that the preference for math as a favorite subject has dependence on a student's gender? Show the analysis and explain your findings.

(b) Does it appear, based on the data in this table, that the preference for social studies as a favorite subject has dependence on a student's gender? Show the analysis and explain your findings.

There is a nice test for dependence that can be applied easily and comes from our formula for conditional probability from the last lesson.

**Exercise #4:** Given that \( P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \), do the following.

(a) If A and B are independent, then rewrite this formula and solve for \( P(A \text{ and } B) \).

(b) The probability that a person is left handed is 12%, the probability they have brown eyes is 42%, and the probability they have brown eyes and are left handed is 2%. Is the event of having brown eyes independent of being left handed? Support your answer.
INDEPENDENT AND DEPENDENT EVENTS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. In each of the following, a scenario is given with two events. Explain whether these events are independent or dependent.
   
   (a) A coin is flipped and lands on a head. The coin is flipped a second time and lands on its head again. Is the probability of it landing on heads the second time dependent on it landing on head the first time? Explain.

   (b) An elementary class consists of 8 boys and 10 girls. A child is chosen at random and it is a girl. A second child is randomly chosen again from the remaining children and it is a boy. Was the probability of choosing the boy dependent on choosing a girl first? Explain.

2. A newspaper did a survey of adults and found that 54% of the population as a whole favored stricter gun control laws. They broke down the results along gender lines and found that 65% of women favored stricter laws while only 44% of men favored them. If a person was selected at random, are the events of being a woman and being in favor of stricter gun control laws dependent or independent? Explain.

3. The eight-sector spinner is back. If the spinner is spun once and the outcome is noted answer the following questions.

   (a) Let the event S be the event of getting a perfect square, i.e. 1 or 4. What is the probability of getting a perfect square, i.e. \( P(S) \)?

   (b) Let E be the event of getting an even. What is the probability of getting a perfect square given you got an even, i.e. \( P(S \mid E) \)? Are the two events independent or dependent? Explain.

   (c) Let M be the event of getting a multiple of four. What is the probability of getting a perfect square given that you got a multiple of four, i.e. \( P(S \mid M) \)? Are the two events independent or dependent? Explain.
4. If two events, A and B, and independent then $P(A \text{ and } B) =$

\[
\frac{P(A)}{P(B)} \quad \text{or} \quad \frac{P(B)}{P(A)}
\]

(1) $P(A) \cdot P(B)$  \hspace{1cm}  (3) $P(B) \cdot P(A)$

(2) $P(A) + P(B)$  \hspace{1cm}  (4) $P(A) + P(B)$

5. There is a 34% chance that a person picked at random from the adult population is a regular smoker of cigarettes and an 18% chance that a person picked has emphysema. If the percent of the adult population that are both regular smokers and suffer from emphysema is 14%, is being a smoker independent from having emphysema? Justify your result by using the **Product Test for Independence**.

6. The two-way frequency table below shows the proportions of a population that have given hair color and eye color combinations. Use this table to answer the following.

(a) Show that the events of having green eyes and red hair are dependent.

<table>
<thead>
<tr>
<th>Eye Color</th>
<th>Blue</th>
<th>Blond</th>
<th>Red</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>0.17</td>
<td>0.21</td>
<td>0.02</td>
<td>0.40</td>
</tr>
<tr>
<td>Brown</td>
<td>0.21</td>
<td>0.13</td>
<td>0.01</td>
<td>0.35</td>
</tr>
<tr>
<td>Green</td>
<td>0.07</td>
<td>0.03</td>
<td>0.15</td>
<td>0.25</td>
</tr>
<tr>
<td>Total</td>
<td>0.45</td>
<td>0.37</td>
<td>0.18</td>
<td>1.00</td>
</tr>
</tbody>
</table>

(b) Many of the hair colors have dependence on eye color. Does having blond hair have a dependence on having brown eyes? Show the analysis that leads to your decision.

7. The month of March has 31 days in it. In New York, March has days when it snows, days when it rains, and days when it does both. This breakdown is shown in the Venn diagram below.

Based on the diagram, are the events of having snow and having rain dependent or independent? Justify.
MULTIPLYING PROBABILITIES
COMMON CORE ALGEBRA II

Probabilities involving **single-stage experiments** are easy enough because only one thing is happening to affect the probability, i.e. you flip a coin once, you pick one person at random, you pull one card out of a deck. Probabilities, both empirical and theoretical, become increasing more complicated with **multi-stage experiments**, where more than one thing happens, i.e. you flip a coin three times. How we handle these types of probabilities actually comes from the conditional probability formula.

**Exercise #1:** Given that the probability of event B occurring given event A has occurred is

\[ P(B | A) = \frac{P(A \text{ and } B)}{P(A)} \]

answer the following.

(a) Rewrite this formula, solving for \( P(A \text{ and } B) \). 

(b) How could you write this formula if events A and B were **independent**?

This rearrangement of the conditional probability formula gives us a useful tool for calculating the probability of events that occur in **multi-stage experiments**. You will easily be able to accomplish this if you systematically phrase the questions as unions of events (events connected by AND).

**Exercise #2:** Consider the spinner shown below. The spinner is spun twice and the result is recorded.

(a) Are the outcomes of the two spins dependent or independent?

(b) What is the probability that you will get an even on the first spin and a number greater than five on the second spin?

(c) What is the probability that you will spin a prime number and a perfect square (in either order)? Note that this is more complex than (b).
As experiments grow more complicated with more stages, theoretical probability becomes increasingly more complicated. It is especially important to note whether you are sampling **without or without replacement**.

**Exercise #3:** A class consists of 12 girls and 8 boys. A group of three is picked to give a speech. If the students are picked at random, what is the probability that they all will be boys? Use the events below to show how you calculated your final answer.

Let: $E_1 =$ Event that the first picked was a boy  
$E_2 =$ Event that the second picked was a boy  
$E_3 =$ Event that the third picked was a boy

The **multiplication property of probability** is crucial in many applications in engineering decision making.

**Exercise #4:** Say that a power generating facility has three primary safety switches in case of an emergency. The probability that any one of these switches would fail is 5%. What is the probability all three will fail given that the switches are **independent** of one another?

Many times when using the multiplication rule we need to be careful about how we frame the question. But, if we properly frame it in terms of AND and OR logical connectors, then the rules of probability will work out.

**Exercise #5:** A company was determining the effectiveness of its warranty sales on computers. They took data on the number of customers who purchased warranties on two different brands of computers. If a customer was chosen at random, what is the probability they did not purchase a warranty?

<table>
<thead>
<tr>
<th>Type</th>
<th>Percentage of Customers Purchasing</th>
<th>Percent of Those Who Purchased that Also Purchased Warranty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>68%</td>
<td>35%</td>
</tr>
<tr>
<td>Type 2</td>
<td>32%</td>
<td>56%</td>
</tr>
</tbody>
</table>
MULTIPLYING PROBABILITIES
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. A fair coin is flipped four times. Find:
   (a) The probability it will land up heads each time. (b) The probability it will land the same way each time (slightly different from (a)).

2. A first grade class of nine girls and seven boys walks into class in alphabetical order (by last name). What is the probability that three girls are the first to enter the room? Show your calculation.
   (1) 0.15 (3) 0.35
   (2) 0.20 (4) 0.45

3. A bag of marbles contains 12 red marbles, 8 blue marbles, and 5 green marbles. If three marbles are pulled out, find each of the following probabilities. In each we specify either replacement (the marbles go back into the bag after each pull) or no replacement.
   (a) Find the probability of pulling three green marbles out with replacement. (b) Find the probability of pulling out 3 red marbles without replacement.
   (c) Find the probability of pulling out 3 marbles of the same color without replacement. This is more complex than the other two.
   (d) Find the probability of pulling out two blue marbles and one green marble in any order with replacement. Be careful as there are multiple ways this can be done that will add.
4. The table below shows the percents of graduating seniors who are going to college, broken down into subgroups by gender. If a student was picked at random find the probability that:

(a) They would be a female going to college.

(b) They would be a male not going to college.

(c) They would be going to college.

(d) They would not be going to college.

<table>
<thead>
<tr>
<th>Percent of Graduating Seniors</th>
<th>Percent of Subgroup Going to College</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>46%</td>
</tr>
<tr>
<td>Female</td>
<td>54%</td>
</tr>
</tbody>
</table>

5. If a safety switch has a 1 in 10 chance of failing, how many switches would a company want to install in order to have only a 1 in one million chance of them all failing at the same time? Show your reasoning.

**Reasoning**

6. If the probability of winning a carnival game was \( \frac{2}{5} \) and Max played it five times, write an expression that would calculate the probability he won the first three games and lost the last two. Use exponents to express your final answer, but do not evaluate.
UNIT #13

STATISTICS

Lesson #1 – Variability and Sampling
Lesson #2 – Population Parameters
Lesson #3 – The Normal Distributions
Lesson #4 – The Normal Distribution and Z-Scores
Lesson #5 – Sample Means
Lesson #6 – Sample Proportions
Lesson #7 – The Difference in Samples Means
Lesson #8 – Linear Regression and Lines of Best Fit
Lesson #9 – Other Types of Regression
Data is everywhere. It's in our newspapers, it's in our science classes, it shows up in economics, medicine and anywhere else that variability occurs. Variability is simply the property of outcomes being different. The tools of statistics are designed to explain this variability.

There are many types of variability. It is good to understand these sources in order to minimize the ones that we are not studying.

Exercise #1: The following types of variability can change uniformity of a data set. For each give an example from any field.

(a) **Observational or Measurement Variability**: Variability that is introduced due to either our measuring instruments not being precise enough or differences in how two different people read the measurement.

(b) **Natural Variability or Inter-Individual Variability**: Variability that accounts for the fact that members of a populations are simply different.

(c) **Induced Variability**: This type of variability is in marked contrast to natural. It occurs because we have assigned our population or sample to two or more treatment groups and then observe the variability between the groups.

(d) **Sample Variability**: This is the type of variability that occurs when we take multiple samples from a population randomly. These samples will be different due to the randomness of the sampling process.

Remember, through all of our work in this unit, we are really trying to explain the variability of data within either a population or a sample and then using this to determine if the variability can be attributed to one of the factors above to the exclusion of the others.
There are many different situations in which we collect data. They have important differences and all of them depend on randomization in one way or another.

**Exercise #2:** The three major types of ways to collect data are described below. Give an example of each and explain how randomization is part of each method. Randomization is used primarily to eliminate variability caused by some type of bias.

(a) **Surveys:** Collections of data from a population where variability is not induced by treatments but by the sample itself (sampling variability).

(b) **Observational Studies:** Collections of data from a population where assignment of individuals from the population into treatment groups is not under the control of those performing the study.

(c) **Experimental Studies:** In experimental studies individuals are assigned randomly to treatment groups in order to determine the effect of the treatment on the variability of the data. In these cases, the assignment, although random, is under the control of those performing the study.

Random sampling is critical for being able to minimize variability due to sampling bias. Random sampling can be done using a variety of different techniques. Simple random sampling can be accomplished using a random number table.

**Exercise #3:** A list of 10 people's heights, in inches, is shown below.

<table>
<thead>
<tr>
<th>Person #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>70</td>
<td>68</td>
<td>60</td>
<td>75</td>
<td>65</td>
<td>69</td>
<td>58</td>
<td>62</td>
<td>66</td>
<td>63</td>
</tr>
</tbody>
</table>

(a) Randomly select five heights from this list by using the random number table that goes with this lesson. Choose a random spot in the table and move down the column. Select the first digit of each number. If you get a repeat, eliminate and keep going. If you get a 0, use this as the 10.

(b) Calculate the sample mean to the nearest tenth. Compare to others in the class. What type of variability is being introduced through this process?
APPLICATIONS

1. Scientists randomly select ten groups from a population of men over 50 years old. They calculate the mean weights of each of these groups. The variability between these means can be best attributed to

   (1) measurement variability  (3) induced variability
   (2) natural variability      (4) sampling variability

2. Max and Daniel are measuring the amount of time it takes for a ball to roll down a ramp at different heights. For each trial, both Max and Daniel take turns rolling the ball and working the stop watch. They do this in order to quantify which of the following sources of variability?

   (1) measurement variability  (3) induced variability
   (2) natural variability      (4) sampling variability

3. Which of the following scenarios would be an attempt to quantify induced variability?

   (1) a phone survey of political preferences during election season
   (2) multiple random samples of products from an assembly line to check for defects
   (3) random assignment of people to a control group and a group taking a drug to lower cholesterol
   (4) recording the variability in the measurement of a soil sample's weight by the same machine

4. Which of the following research questions would involve collecting data through a survey?

   (1) Watching people exit a grocery store to see the percent who use reusable bags.
   (2) Assigning people to two groups to see the effect of a particular amount of sleep.
   (3) Calling people on the telephone to see if they will be voting in the upcoming election.
   (4) Dropping salt cubes into two different liquids to determine which dissolves faster.

5. In which of the following cases would an observational study be necessary as compared to an experiment study?

   (1) The study of how increased nutrient levels affect plant growth.
   (2) The study of how educational levels affect median household income.
   (3) The study of how a vaccine affects the percent of mice that get a particular disease.
   (4) The study of how noise level affects the sleep patterns of volunteers in a sleep study.
6. In an experimental study, a lab wanted to divide volunteers into two groups to determine the effect of a particular phone app to help make people more punctual (on time). The 50 volunteers in the study will be assigned to either a group of 25 who use the app for a week or a group of 25 who do not use the app. The participants were asked to come to a lab to receive the app (or not) at 10:00 am on a Monday. Answer the following questions:

(a) Why would those performing the study not want to assign the participants in the two treatments (groups) based on who showed up to the study session first?

(b) Propose a way to use a random number table to generate a simple random selection that eliminates the bias that you discussed in part (a).

7. Two groups of subjects were divided in an experimental study. One group was given a drug to help speed up their metabolism and result in weight loss. The other group was given a placebo (a pill that looks identical to the one given to the other group, but without the weight loss drug). After a month of the experiment, the weight loss of each individual in each of the two groups were measured. In general, people in the group who took the metabolism drug did lose more weight, although there were differences in the amount each lost. There are two main types of variability occurring in this study. Describe each type below in the context of this study.

Induced Variability
Natural Variability

REASONING

8. If you were trying to conduct a survey of political preferences for likely voters in an upcoming election and decided to dial 1,000 randomly generated land-line phone numbers (not cell), why might this still introduce bias into the sampling?
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<td>89628</td>
<td>99681</td>
<td>41047</td>
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<td>88642</td>
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</tbody>
</table>
When we conduct a study, the complete set of all subjects that share a common characteristic that is being studied is known as the **population**. All populations have **natural or inter-individual variability**. Most of the time, the entire population is not measured, but a sample is taken to infer characteristics of a population. Still, all populations in theory have **population parameters** that describe the population, such as its mean, standard deviation, and interquartile range.

**Exercise #1**: 18 students in Mr. Weiler's Advanced Calculus class took a quiz with the following results in ascending order.

56, 68, 72, 72, 75, 78, 80, 84, 84, 85, 88, 88, 90, 93, 95, 99, 100, 100

(a) Use your calculator to determine the mean, the median, and the quartiles for this data set. Then, construct a simple box-and-whiskers (box plot) for this data set.

(b) What is the interquartile range of this data set? In theory, what percent of the data set should lie between the first and third quartiles? Is that true for this data set?

(c) What is the population standard deviation for this data set to the nearest tenth? How do you interpret the standard deviation?

(d) What percent of the scores were within one standard deviation of the mean? Within two standard deviations of the mean? Round your percents to the nearest percent and show your work.

| Within One Standard Deviation of the Mean | Within Two Standard Deviations of the Mean |
Sometimes data is grouped in a frequency chart. We still should be able to calculate the basic population parameters when the information is given in this form.

**Exercise #2:** A small company has salaries for their 50 employees as given in the table below

<table>
<thead>
<tr>
<th>Salary ($x_i$)</th>
<th>Frequency ($f_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25,000</td>
<td>5</td>
</tr>
<tr>
<td>32,000</td>
<td>21</td>
</tr>
<tr>
<td>45,000</td>
<td>14</td>
</tr>
<tr>
<td>58,000</td>
<td>7</td>
</tr>
<tr>
<td>75,000</td>
<td>2</td>
</tr>
<tr>
<td>120,000</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Find the mean and standard deviations of the salary range.

(b) What is the median of this data set? Why is the median considerably lower than the mean in this data set?

(c) Does more or less than 50% of the data set fall within one standard deviation of the mean? Show the analysis that leads to your answer.

Although we have often concentrated on experimental studies where data is collected and means are found, many times we use statistics to represent results of a survey where we are interested in what proportion of a population share a certain characteristic. These proportions are most expressed as decimals, but sometimes are represented by fractions or percents.

**Exercise #4:** A questionnaire went home to all juniors concerning their ability to bring and use mobile devices at school. The questionnaires constituted a *census* since all of the juniors were surveyed. Of the 742 juniors, 564 of them reported having web-enabled mobile devices. What was the population proportion for web-enabled devices? Express your answer as a decimal and as a percent.

**Exercise #5:** The proportion of eggs that get cracked in a local egg handling facility is 0.023. If 2,500 dozen eggs are packaged in the factor per day, what should we expect to be the number of eggs cracked per day?

(1) 350  (3) 230

(2) 450  (4) 690


**FLUENCY**

1. Which of the following formulas, written in summation notation, would represent the mean of the data set \( \{x_1, x_2, ..., x_n\} \)? Explain your choice.

   \[
   \text{(1)} \quad \frac{1}{n} \sum_{i=1}^{n} x_i \\
   \text{(2)} \quad \frac{1}{n} \sum_{i=1}^{n} x_i^2 \\
   \text{(3)} \quad n \sum_{i=1}^{n} x_i \\
   \text{(4)} \quad \sum_{i=1}^{n} x_i
   \]

2. The standard deviation of a population characteristics measures

   (1) The difference between the maximum and minimum values.
   (2) The difference between the third quartile and first quartile values.
   (3) The average distance a data value is away from the mean.
   (4) The average distance a data value is away from the median.

3. The interquartile range of the data set \( \{4, 7, 10, 13, 18, 22, 30\} \) is

   (1) 15  
   (2) 18  
   (3) 7   
   (4) 10

**APPLICATIONS**

4. If 348 freshmen out of 622 have cell phones, then the population proportion, \( p \), for freshmen cell phone ownership is

   (1) 0.56  
   (2) 0.35  
   (3) 0.72  
   (4) 0.44

5. If a population has 824 subjects, then about how many would have characteristics in the upper quartile?

   (1) 412  
   (2) 280  
   (3) 368  
   (4) 206
6. A school is tracking its freshmen attendance for the first marking period. Shown below is a table summarizing their findings for the 284 members of the freshmen class.

(a) Find the mean and median number of days absent. Round your mean to the nearest tenth.

<table>
<thead>
<tr>
<th>Days Absent ((x_i))</th>
<th>Number of Students ((f_i))</th>
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<tbody>
<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>64</td>
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<tr>
<td>2</td>
<td>18</td>
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<td>3</td>
<td>22</td>
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<td>4</td>
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<td>7</td>
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<td>6</td>
<td>8</td>
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<tr>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) What is the population standard deviation for this data set? Round to the nearest tenth.

(c) What proportion of the population that has an absenteeism greater than 4 days?

7. The heights of the 15 players on the Arlington boys' varsity basketball team are given below in inches.

66, 67, 68, 68, 70, 72, 72, 73, 74, 75, 75, 75, 76, 77, 79

(a) Find the mean and standard deviation of this data set. Use the population standard deviation. Round both to the nearest tenth.

(b) Determine the proportion of the population that falls within one standard deviation and within two standard deviations of the mean. State your values in decimal form.

One standard deviation from the mean:

Two standard deviations from the mean:

(c) Use the random number table for this lesson to pick a random sample of five players from this list. Do this by picking a random two digit column along the page. Scan down the column until you have picked 5 random integers that fall from 1 to 15. Write down your sample and calculate its mean.
THE NORMAL DISTRIBUTION
COMMON CORE ALGEBRA II

Many populations have a distribution that can be well described with what is known as The Normal Distribution or the Bell Curve. This curve, as seen in the accompanying handout to this lesson, shows the percent or proportion of a normally distributed data set that lies certain amounts from the mean.

Exercise #1: For a population that is normally distributed, find the percentage of the population that lies
(a) within one standard deviation of the mean.                      (b) within two standard deviations of the mean.

(c) more than three standard deviations away from the mean.    (d) between one and two standard deviations above the mean.

As can be easily seen from Exercise #1, the majority of any normally distributed population will lie within one standard deviation of its mean and the vast majority will lie within two standard deviations. A whole variety of problems can be solved if we know that a population is normally distributed.

Exercise #2: At Arlington High School, 424 juniors recently took the SAT exam. On the math portion of the exam, the mean score was 540 with a standard deviation of 80. If the scores on the exam were normally distributed, answer the following questions.
(a) What percent of the math scores fell between 500 and 660?
(b) How many scores fell between 500 and 660? Round your answer to the nearest whole number.

(c) If Evin scored a 740 on her math exam, what percent of the students who took the exam did better than her?
(d) Approximately how many students did better than Evin?
**Exercise #3:** The heights of 16 year old teenage boys are normally distributed with a mean of 66 inches and a standard deviation of 3. If Jabari is 72 inches tall, which of the following is closest to his height’s percentile rank?

1. 85th  
2. 67th  
3. 98th  
4. 93rd

**Exercise #4:** The amount of soda in a standard can is normally distributed with a mean of 12 ounces and a standard deviation of 0.6 ounces. If 250 soda cans were pulled by a company to check volume, how many would be expected to have less than 11.1 ounces in them?

1. 17  
2. 23  
3. 28  
4. 11

**Exercise #5:** Biologists are studying the weights of Red King Crabs in the Alaskan waters. They sample 16 crabs and compiled their weights, in pounds, as shown below.

9.8, 10.1, 11.1, 12.4, 11.8, 13.2, 12.8, 12.5, 13.7, 11.6, 13.4, 12.3, 12.6, 14.8, 14.2, 15.1

(a) Determine the mean and sample standard deviation for this sample of crabs. Round both statistical measures to the nearest tenth of a pound.

(b) Why does this sample indicate that the population would be well modeled using a normal distribution? Explain. Hint – Use your calculator to sort this data in ascending order.

(c) Assuming your mean and standard deviation from part (a) apply to a normally distributed population of crabs caught in Alaska, what percent will fall between 9.6 pounds and 15.6 pounds?

(d) If fishermen must throw back any crab caught below 10.4 pounds, approximately what percent of the crabs caught will need to be thrown back if the weights are normally distributed?
THE NORMAL DISTRIBUTION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. A variable is normally distributed with a mean of 16 and a standard deviation of 6. Find the percent of the data set that:

(a) is greater than 16  (b) falls between 10 and 22  (c) is greater than 28

(d) is less than 1  (e) falls between 4 and 19  (f) falls between 22 and 31

APPLICATIONS

2. The weights of Siamese cats are normally distributed with a mean of 6.4 pounds and a standard deviation of 0.8 pounds. If a breeder of Siamese cats has 128 in his care, how many can he expect to have weights between 5.2 and 7.6 pounds?

(1) 106  (3) 98
(2) 49  (4) 111

3. If one quart bottles of apple juice have weights that are normally distributed with a mean of 64 ounces and a standard deviation of 3 ounces, what percent of bottles would be expected to have less than 58 ounces?

(1) 6.7%  (3) 0.6%
(2) 15.0%  (4) 2.3%

4. Historically daily high temperatures in July in Red Hook, New York, are normally distributed with a mean of 84°F and a standard deviation of 4°F. How many of the 31 days of July can a person expect to have temperatures above 90°F?

(1) 6  (3) 9
(2) 2  (4) 4
5. The weights of four year old boys are normally distributed with a mean of 38 pounds and a standard deviation of 4 pounds. Which of the following weights could represent the 90th percentile for the weight of a four year old?

(1) 47 pounds  (3) 43 pounds
(2) 45 pounds  (4) 41 pounds

6. The lengths of songs on the radio are normally distributed with a mean length of 210 seconds. If 38.2% of all songs have lengths between 194 and 226 seconds, then the standard deviation of this distribution is

(1) 16 seconds  (3) 8 seconds
(2) 32 seconds  (4) 64 seconds

7. The heights of professional basketball players are normally distributed with a standard deviation of 5 inches. If only 2.3% of all pro basketball players have heights above 7 foot 5 inches, then which of the following is the mean height of pro basketball players?

(1) 6 feet 5 inches  (3) 6 feet 10 inches
(2) 6 feet 2 inches  (4) 6 feet 7 inches

8. On a recent statewide math test, the raw score average was 56 points with a standard deviation of 18. If the scores were normally distributed and 24,000 students took the test, answer the following questions.

(a) What percent of students scored below a 38 on the test?
(b) How many students scored less than a 38?

(c) If the state would like to scale the test so that a 90% would correspond to a raw score that is one and a half standard deviations above the mean, what raw score is needed for a 90%?

(d) How many of the 24,000 students receive a scaled score greater than a 90%?

(e) The state would like no more than 550 of the 24,000 students to fail the exam. What percent of the total does the 550 represent? Round to the nearest tenth of a percent.

(f) What should the raw passing score be set at so that no more than the 550 students fail?
THE NORMAL DISTRIBUTION

BASED ON STANDARD DEVIATION
The normal distribution can be used in increments other than half-standard deviations. In fact, we can use either our calculators or tables to determine probabilities (or proportions) for almost any data value within a normally distributed population, as long as we know the population mean, $\mu$, and the population standard deviation, $\sigma$. But, first, we will introduce a concept known as a data value’s z-score.

**THE Z-SCORE OF A DATA VALUE**

For a data point $x_i$, its z-score is calculated by: $z = \frac{x_i - \mu}{\sigma}$. It calculates how far from the mean, in terms of standard deviations, a data point lies. It can be positive if the data point lies above the mean or negative if the data point lies below the mean.

**Exercise #1:** Boy’s heights in seventh grade are normally distributed with a mean height of 62 inches and a standard deviation of 3.2 inches. Find z-scores, rounded to the nearest hundredth, for each of the following heights. Show the calculation that leads to your answer.

(a) $x_i = 66$ inches
(b) $x_i = 57$ inches
(c) $x_i = 70$ inches

Z-scores give us a way to compare how far a data point is away from its mean in terms of standard deviations. We should be able to compute a z-score for a data value and go in the opposite direction.

**Exercise #2:** Jeremiah took a standardized test where the mean score was a 560 and the standard deviation was 45. If Jeremiah’s score resulted in a z-value of 1.84, then what was Jeremiah’s score to the nearest whole number?

With z-scores, we can then determine the probability that a subject picked from a normally distributed population would have a characteristic in a certain range. Z-score tables come in many different varieties. The one that comes with this lesson shows only the right hand side, so symmetry will have to be used to determine probabilities.

**Exercise #3:** The lengths of full grown sockeye salmon are normally distributed with a mean of 29.2 inches and a standard deviation of 2.4 inches.

(a) Find z-scores for sockeye salmon whose lengths are 25 inches to 32 inches. Round to the nearest hundredth.

(b) Use the z-score table to determine the proportion of the sockeye salmon population, to the nearest percent, that lies between 25 inches and 32 inches. Illustrate your work graphically.
Exercise #4: If the scores on a standardized test are normally distributed with a mean of 560 and a standard deviation of 75. Answer the following questions by using z-scores and the normal distribution table.

(a) Find the probability that a test picked at random would have a score larger than 720. Round to the nearest tenth of a percent.  
(b) Find the probability that a completed test picked at random would have a score less than 500. Round to the nearest tenth of a percent.  
(c) Find the probability that a completed test picked at random would have a score between 500 and 600.  
(d) Find the probability that a completed test picked at random would have a score between 600 and 700.

This process is sometimes used to determine a particular data point’s percentile, which is the percent of the population less than the data point.

Exercise #5: The average weight of full grown beef cows is 1470 pounds with a standard deviation of 230 pounds. If the weights are normally distributed, what is the percentile rank of a cow that weighs 1,750 pounds?

(1) 89<sup>th</sup>  
(2) 76<sup>th</sup>  
(3) 49<sup>th</sup>  
(4) 35<sup>th</sup>

Your graphing calculator can also find these proportions or percent values. Each calculator’s inputs and language will be slightly different, although many will do much of the work for you, even allowing you to not think about the z-scores.

Exercise #6: Given that the volume of soda in a 12 ounce bottle from a factory varies normally with a mean of 12.2 ounces and a standard deviation of 0.6 ounces, use your calculator to determine the probability that a bottle chosen at random would have a volume:

(a) Greater than 13 ounces.  
(b) Less than 11 ounces  
(c) Between 11.5 and 12.5 ounces
THE NORMAL DISTRIBUTION AND Z-SCORES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. A population has a mean of $\mu = 24.8$ and a standard deviation of $\sigma = 4.2$. For each of the following data values, calculate the z-value to the nearest hundredth. You do not need to read the Normal table.

   (a) $x_i = 30$
   (b) $x_i = 35$
   (c) $x_i = 19$
   (d) $x_i = 15.4$
   (e) $x_i = 24.8$
   (f) $x_i = 33.2$

2. A population has a mean of 102.8 and a standard deviation of 15.4. If a data point has a z-value of 1.87 then which of the following is the value of the data point?

   (1) 28.8
   (2) 86.7
   (3) 131.6
   (4) 152.3

APPLICATIONS

Get practice with both the Normal Distribution Table and your calculator when doing the following problems.

3. A recent study found that the mean amount spent by individuals on a music service website was normally distributed with a mean of $384 with a standard deviation of $48. Which of the following gives the proportion of the individuals that spend more than $400?

   (1) 0.43
   (2) 0.74
   (3) 0.12
   (4) 0.37

4. The hold time experienced by people calling a government agency was found to be normally distributed with a mean of 12.4 minutes and a standard deviation of 4.3 minutes. Which percent below represents the percent of calls answered in less than 5 minutes?

   (1) 4.3%
   (2) 5.3%
   (3) 6.8%
   (4) 12.9%
5. The national average price per gallon for gasoline is normally distributed with a mean (currently) of $2.34 per gallon with a standard deviation of $0.26 per gallon. Which of the following represents the proportion of the gas prices that lie between $2.00 and $3.00?

(1) 56%  (3) 84%
(2) 72%  (4) 90%

6. If the average teacher salary in the United States is $45,753 and salaries are normally distributed with a standard deviation of $7890, would a salary of $40,000 per year be in the lowest quintile of teacher salary? (Do a quick Internet search on the term quintile if you don't know what it means).

7. The average rent for a one bedroom apartment (in the Winter of 2015) in New York City is a whopping $2801 per month with a standard deviation of $920.

(a) If rents are normally distributed, what percent of the apartments will be less than $2,500 per month?

(b) If rents are normally distributed, how realistic is it to believe you will be able to rent a one-bedroom in New York City for less than $1,500 per month? Justify your answer.

(c) A one-bedroom on the Upper East Side with a doorman and views of Central Park was listed at $5,000 per month. How rare is this? Assume the rents are normally distributed.

(d) Do you think the rents are normally distributed? Keep in mind the normal distribution is symmetric about its mean (looks the same on both sides). If it isn't symmetric, what does it look like?

8. A national math competition advances to the second round only the top 5% of all participants based on scores from a first round exam. Their scores are normally distributed with a mean of 76.2 and a standard deviation of 17.1. What score, to the nearest whole number, would be necessary to make it to the second round? To start, look at the table and see if you can determine the z-value that corresponds to the top 5%.
447

STANDARD NORMAL DISTRIBUTION BASED ON Z-VALUE
z
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
2.3
2.4
2.5
2.6
2.7
2.8
2.9
3.0
3.1
3.2
3.3
3.4

0.00

0.01

0.02

0.03

0.04

0.05

0.06

0.07

0.08

0.09

0.0000
0.0398
0.0793
0.1179
0.1554
0.1915
0.2257
0.2580
0.2881
0.3159
0.3413
0.3643
0.3849
0.4032
0.4192
0.4332
0.4452
0.4554
0.4641
0.4713
0.4772
0.4821
0.4861
0.4893
0.4918
0.4938
0.4953
0.4965
0.4974
0.4981
0.4987
0.4990
0.4993
0.4995
0.4997

0.0040
0.0438
0.0832
0.1217
0.1591
0.1950
0.2291
0.2611
0.2910
0.3186
0.3438
0.3665
0.3869
0.4049
0.4207
0.4345
0.4463
0.4564
0.4649
0.4719
0.4778
0.4826
0.4864
0.4896
0.4920
0.4940
0.4955
0.4966
0.4975
0.4982
0.4987
0.4991
0.4993
0.4995
0.4997

0.0080
0.0478
0.0871
0.1255
0.1628
0.1985
0.2324
0.2642
0.2939
0.3212
0.3461
0.3686
0.3888
0.4066
0.4222
0.4357
0.4474
0.4573
0.4656
0.4726
0.4783
0.4830
0.4868
0.4898
0.4922
0.4941
0.4956
0.4967
0.4976
0.4982
0.4987
0.4991
0.4994
0.4995
0.4997

0.0120
0.0517
0.0910
0.1293
0.1664
0.2019
0.2357
0.2673
0.2967
0.3238
0.3485
0.3708
0.3907
0.4082
0.4236
0.4370
0.4484
0.4582
0.4664
0.4732
0.4788
0.4834
0.4871
0.4901
0.4925
0.4943
0.4957
0.4968
0.4977
0.4983
0.4988
0.4991
0.4994
0.4996
0.4997

0.0160
0.0557
0.0948
0.1331
0.1700
0.2054
0.2389
0.2704
0.2995
0.3264
0.3508
0.3729
0.3925
0.4099
0.4251
0.4382
0.4495
0.4591
0.4671
0.4738
0.4793
0.4838
0.4875
0.4904
0.4927
0.4945
0.4959
0.4969
0.4977
0.4984
0.4988
0.4992
0.4994
0.4996
0.4997

0.0199
0.0596
0.0987
0.1368
0.1736
0.2088
0.2422
0.2734
0.3023
0.3289
0.3531
0.3749
0.3944
0.4115
0.4265
0.4394
0.4505
0.4599
0.4678
0.4744
0.4798
0.4842
0.4878
0.4906
0.4929
0.4946
0.4960
0.4970
0.4978
0.4984
0.4989
0.4992
0.4994
0.4996
0.4997

0.0239
0.0636
0.1026
0.1406
0.1772
0.2123
0.2454
0.2764
0.3051
0.3315
0.3554
0.3770
0.3962
0.4131
0.4279
0.4406
0.4515
0.4608
0.4686
0.4750
0.4803
0.4846
0.4881
0.4909
0.4931
0.4948
0.4961
0.4971
0.4979
0.4985
0.4989
0.4992
0.4994
0.4996
0.4997

0.0279
0.0675
0.1064
0.1443
0.1808
0.2157
0.2486
0.2794
0.3078
0.3340
0.3577
0.3790
0.3980
0.4147
0.4292
0.4418
0.4525
0.4616
0.4693
0.4756
0.4808
0.4850
0.4884
0.4911
0.4932
0.4949
0.4962
0.4972
0.4979
0.4985
0.4989
0.4992
0.4995
0.4996
0.4997

0.0319
0.0714
0.1103
0.1480
0.1844
0.2190
0.2517
0.2823
0.3106
0.3365
0.3599
0.3810
0.3997
0.4162
0.4306
0.4429
0.4535
0.4625
0.4699
0.4761
0.4812
0.4854
0.4887
0.4913
0.4934
0.4951
0.4963
0.4973
0.4980
0.4986
0.4990
0.4993
0.4995
0.4996
0.4997

0.0359
0.0753
0.1141
0.1517
0.1879
0.2224
0.2549
0.2852
0.3133
0.3389
0.3621
0.3830
0.4015
0.4177
0.4319
0.4441
0.4545
0.4633
0.4706
0.4767
0.4817
0.4857
0.4890
0.4916
0.4936
0.4952
0.4964
0.4974
0.4981
0.4986
0.4990
0.4993
0.4995
0.4997
0.4998

Area/proportion given in table.

0

z

COMMON CORE ALGEBRA II, UNIT #13 – STATISTICS – LESSON #4
eMATHINSTRUCTION, RED HOOK, NY 12571, © 2015


SAMPLE MEANS
COMMON CORE ALGEBRA II

The vast majority of the statistics that you've done so far has been descriptive. With descriptive statistics, we summarize how a data set "looks" with measures of central tendency, like the mean, and measures of dispersion, like the standard deviation. But, the more powerful branch of statistics is known as inferential where we try to infer properties about a population from samples that we take. We do this by using probability and sampling variability to estimate how likely the sample is given a certain population.

We begin our multi-lesson investigation into inferential statistics with the most basic question. How can we estimate the population mean, \( \mu \), if we know a sample mean, \( x \)? Before answering this question, though, we need to investigate the distribution of sample means.

**Exercise #1:** Say we are investigating the heights of 16 year old American males. Say we know that the population mean height is 65.3 inches with a standard deviation of 4.2 inches. Let's say we take a sample of 16 year old American males. The sample has a size of 30.

(a) Will the mean height of the sample always be 65.3? Why or why not? Could it be significantly different?

(b) Will the standard deviation of the sample be 4.2 inches? Would you expect more or less variation in a sample versus a population?

(c) Run the program NORMSAMP with a mean and standard deviation given above and a sample size of 30. Do at least 200 simulations (500 is preferable). State the minimum and maximum of the sample means, the mean of the sample means, and the standard deviation of the sample means.

- min sample mean = _____________
- Mean of means: _____________
- Standard deviation of means: _____________
- max sample mean = _____________

(d) How does the variability of the sample means compare to the variability of the population? Is it more or less?

(e) State the mean of the sample standard deviations. How does it compare to the population standard deviation? Could you use the standard deviation of a sample to estimate the standard deviation of the population?
We can use our simulation to decide whether a sample could have come from a given population. We can even quantify how likely it would be to happen. This is known as establishing confidence.

**Exercise #2:** Mr. Weiler took a sample of 30 16-year old males and found the mean height of the sample to be 66.4 inches. Do you believe this sample came from this population? Why or why not? Examine the results of your simulation. Quantify how likely this sample (or one greater) was to come from the population simulated.

Strangely enough, this process can be used in order to give a confidence interval for the population mean if we know the sample mean. This is important because in reality the population mean is almost never known and is what we want to infer from the sample mean. The next set of exercises will illustrate how this is done using simulation.

**Exercise #3:** A sample of 50 ripe oranges were taken from a large orchard in order to estimate the mean weight of a ripe orange. The sample mean was 212 grams and the sample standard deviation was 34 grams.

(a) Why does it seem reasonable to use the sample standard deviation as an estimate of the population standard deviation? See Exercise #1(e).

(b) Run a simulation using the 212 as the population mean (even though it is the sample mean) and use 34 as the population standard deviation. State the 5th percentile sample mean and the 95th percentile sample mean.

5th Percentile Sample Mean = _____________
95th Percentile Sample Mean = ____________

(c) Now, try your simulation again, but use the 5th percentile sample mean as the population mean. Where does the 212 lie on the distribution in terms of percentile? Notice how close this is to the 95th percentile.

(d) Now, try your simulation again, but use the 95th percentile sample mean as the population mean. Where does the 212 lie on the distribution in terms of percentile? Notice how close this is to the 5th percentile.

(e) What both (c) and (d) tell us is that by using the 5th and 95th percentile values based on our original sample mean, we have actually found the lowest possible population mean and highest possible population mean that could have resulted in that sample mean 90% of the time. Write the 90% confidence interval below for \( \mu \) based on (b).
SAMPLE MEANS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. The mean of the sample means is

(1) Greater than the population mean.
(2) Less than the population mean.
(3) Equal to the population mean.
(4) Could be greater or less than the population mean.

2. The variation within the sample means is

(1) Less than the variation within the population.
(2) More than the variation within the population.
(3) Equal to the variation within the population.
(4) Could be more or less than the variation within the population.

APPLICATIONS

3. A factory has machines that fill 12 ounce soda bottles repeatedly with an average volume of 12.2 ounces and standard deviation of 0.9 ounces. A new machine was installed and 30 bottles were sampled. It was found that they had an average volume of 11.8 ounces. We want to investigate whether this mean is significantly lower than the original population mean.

(c) Use the sample mean of 11.8 ounces and a standard deviation of 0.9 ounces to generate the 90% confidence interval for the population mean of the new soda filling machine by simulation. Use a sample size of 30 and at least 100 samples to generate your interval. Round you lower and upper estimate for $\mu$ to the nearest hundredth.

$$\mu_L = \underline{\phantom{000}}$$
Interval: $\underline{\phantom{0000}} \leq \mu \leq \underline{\phantom{0000}}$

$$\mu_H = \underline{\phantom{000}}$$

(a) Run NORMSAMP with a mean of 12.2 and a standard deviation of 0.8. Run 100 simulations. What percentile rank would you give the 11.8 ounces (this will vary based on the simulation)?

(b) Based on your findings from (a), can you conclude that this sample mean likely came from the same population or a different population with a lower mean? Explain.

(c) Use the sample mean of 11.8 ounces and a standard deviation of 0.9 ounces to generate the 90% confidence interval for the population mean of the new soda filling machine by simulation. Use a sample size of 30 and at least 100 samples to generate your interval. Round you lower and upper estimate for $\mu$ to the nearest hundredth.
Let's work some now with the **95% confidence interval**. It will be easiest to use 200 simulations to generate this interval as we will soon see.

4. Consider a normal distribution with 95% of the probability (or distributions) **centered about the population mean** $\mu$.

   (a) What percent lies in half of the shaded area shown in the diagram?

   (b) Explain why 2.5% of the population must lie in each of the un-shaded areas of the graph.

   (c) What is 2.5% of 200?

5. Researchers have found that the average number of hours per week spent by adults watching television is 34.2 with a standard deviation of 6.8 hours. Researchers wanted to determine if there was an effect to sampling people with tablets. They found that a random sample of 40 people with tablets had a sample mean of 36.3 hours per week with a sample standard deviation of 5.4 hours.

   (a) Run NORMSAMP with a population mean of 34.2 hours and a standard deviation of 6.8 hours. Do 200 simulations. This will take some time (approximately 8 minutes). Based on your results, what is the approximate percentile rank of 36.3 (remember it is out of 200 now)?

   (b) Do you have significant evidence that the 36.3 comes from a population with a mean that is higher than 34.2? Explain your thinking.

   (c) Now, let's attempt to construct the **95% confidence interval** for the sample whose mean was 36.3. Run a simulation with a mean of 36.3, a standard deviation of 5.4, a sample size of 40, and with 200 simulations. Find the 2.5th percentile as the lower limit and the 97.5th percentile as the upper limit.

   $$\mu_L = \text{___________}$$

   $$\mu_U = \text{___________}$$

   Interval: \( \mu \leq \text{___________} \leq \text{___________} \)

   (d) The **theoretical** (versus simulated) 95% confidence interval can be found using the formula below, where $\overline{x}$ is the observed sample mean and $s$ is the sample standard deviation. Use this formula and compare to the interval from above.

   $$\overline{x} - 2 \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \overline{x} + 2 \cdot \frac{s}{\sqrt{n}}$$

   Interval: \( \mu \leq \text{___________} \leq \text{___________} \)
COMMON CORE ALGEBRA II

SAMPLE PROPORTIONS

Many times we are interested in determining a confidence interval for the population mean, \( \mu \), based on a sample mean, \( \bar{x} \). Sometimes, though, we want to simply know what proportion, \( p \), of a population shares a certain characteristic. We again infer characteristics about \( p \) based on the proportion of a sample, \( \hat{p} \) (p “hat” as it is often called).

Exercise #1: A school is trying to determine the proportion of students who own cell phones. They do a survey of all juniors and find that 168 out of 236 of have cell phones. They then take a sample of freshmen and find that 30 out of 52 freshmen in the sample own cell phones.

(a) Calculate the population proportion, \( p \), of juniors who own cell phones. Round to the nearest hundredth.

(b) Calculate the sample proportion, \( \hat{p} \), of freshmen who own cell phones. Round to the nearest hundredth.

Clearly, in the last example, the sample proportion of freshmen who own cell phones is less than the population proportion of juniors who own cell phones. But, can we attribute that variability to the two “treatments”, i.e. juniors versus freshmen, or could the variability be due to sampling variability, i.e. the random chance that we just picked a group of freshmen who have an unusually low rate of cell phone ownership? We can establish how likely this is to happen by using simulation.

Exercise #2: We would like to determine how likely it is that a sample of 52 out of a population with a proportion of cell phone ownership of 71% or 0.71 would result in a sample proportion of 0.58.

(a) Run the program PSIMUL with a \( p \) value of 0.71 and a sample size of 52 for 100 simulations. How many of the 100 simulations had a proportion less than or equal to 0.58?

(b) Based on your answer to (a), how likely is it that a sample of 52 from a population with a cell phone ownership of 71% would result in a sample proportion of only 0.58 or less?

(c) Is it possible that a sample of 52 from a population with a cell phone ownership rate of 71% could have a sample proportion of 0.58 or less? Justify your answer.

(d) What conclusion can you make about freshmen cell phone ownership compared to ownership by juniors? Explain.
**Inferential statistics** is never about proving beyond **any doubt** that a sample either can or cannot come from a certain population. It is about **quantifying how likely it is that it could come from a given population**. Let’s continue exploring this question of sample proportions.

**Exercise #3:** Let’s say we have a population with a 0.25 proportion of being 65 years or older. Let’s take different sized samples from this population and see how the sample proportions behave. Use the program **PSIMUL** to simulate a population with a proportion of 0.25 for various sample sizes and 100 simulations.

(a) Fill in the table below.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Low to High ( \hat{p} ) values</th>
<th>Range in ( \hat{p} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) What was the effect of increasing the sample size on the sample proportions that were simulated? Why does this make sense?

(c) Run PSIMUL one more time with a sample size of 50 but for 200 simulations. Using your results, find the value that represents the 5\(^{th}\) percentile of \( \hat{p} \) values. Find the result that represents that 95\(^{th}\) percentile of the \( \hat{p} \) values. Then, write the **90\% confidence interval** for this sample size coming from this population.

(d) If researchers surveyed 50 people walking out of a movie and found that 21 of them were 65 years or older, do you believe this sample came from the general population with a \( p = 0.25 \)? Why or why not.
SAMPLE PROPORTIONS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. Historically, the proportion of emperor penguins with adult weights above 60 pounds is 0.64. Take this to be the population proportion for this characteristic.
   
   (a) A sample of 26 emperor penguins in a zoo found that 20 of the penguins had adult weights above 60 pounds. Calculate the sample proportion, \( \hat{p} \), for this sample.
   
   (b) Run PSIMUL with a population proportion of \( p = 0.64 \) with a sample size of 26. Do 100 simulations. What percent of these simulations resulted in a \( \hat{p} \) value at or above what you found in part (a)?
   
   (c) Do you have enough evidence from (b) to conclude that penguins raised in a zoo have a significantly higher proportion of weights above 60 pounds? Why or why not?

2. Let’s stick with our emperor penguins from #1. Out of a sample of 56 penguins from a zoo, it was found that 43 penguins had weights over 60 pounds. Run PSIMUL again, but now with a sample size of 56. Continue to use \( p = 0.64 \) and 100 simulations. Do you now have stronger evidence that penguins raised in zoos have a higher proportion with weights over 60 pounds? Explain.

3. In general, as sample size increases, the range in the distribution of sample proportions
   
   (1) increases  (3) decreases
   
   (2) stays the same  (4) could increase or decrease

4. In a population with a proportion \( p = 0.35 \), if samples of size 30 were repeatedly taken, then we would expect approximately 90% of those samples proportions to fall within which of the following ranges?
   
   (1) 0.28 to 0.42  (3) 0.21 to 0.49
   
   (2) 0.18 to 0.54  (4) 0.31 to 0.39
5. A sample of the graduating high school class was questioned about their plans after college. We worked with this sample of graduating seniors in our unit on probability. The two-way frequency chart below summarizes the results of the questionnaire. The school would like to investigate the effect of gender on the rate that students go to college.

<table>
<thead>
<tr>
<th>Gender</th>
<th>Going to College</th>
<th>Not Going to College</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>16</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td>Female</td>
<td>13</td>
<td>9</td>
<td>22</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>29</strong></td>
<td><strong>23</strong></td>
<td><strong>52</strong></td>
</tr>
</tbody>
</table>

(a) Calculate the sample proportion of students going to college for the subgroups male and female. Round to the nearest hundredth.

\[
\hat{p}(\text{males}) = \frac{\text{number of males going to college}}{\text{total number of males}} \\
\hat{p}(\text{females}) = \frac{\text{number of females going to college}}{\text{total number of females}}
\]

The proportion for females going to college is higher than for males going to college (a greater percentage of females go to college than males). Is this due to induced variability or sampling variability? This boils down to asking if the difference is statistically significant.

(b) Design a simulation that would test how likely it is for a sample of 30 (the number of men) from a population that has the \( \hat{p}(\text{female}) \) would result in the \( \hat{p}(\text{male}) \) or below. Explain the simulation and what results you found.

(c) Based on the table above, would you conclude that the overall population proportion of females going to college is greater than the proportion of males? If you believe you have enough evidence from your simulation, explain why. If you do not believe you do, also explain.

(d) Why could this study be an example of both a sample survey and an observational study? Look back at their definitions from the first lesson to fully explain your answer. Also explain how the types of variability introduced.
In a classic experiment two or more treatment groups are randomly created and then subjected to the different treatments. The question then is how to determine if the variability seen between the two groups is due to the treatment.

**Exercise #1:** Suppose 50 people were chosen to try out a new diet pill to help increase weight loss. The people are randomly divided into two groups. One is given the pill while the other is given a placebo (a pill designed to look like real one, but with no medicine). There are two main ways variability can be introduced to the results. Discuss each.

**Induced** (Variability created because of the treatment the subject was placed in):

**Natural** (Variability just because people, animals, plants, etcetera, are naturally different):

The question, then, is how we can distinguish between the two types. Let's look at a case study.

**Exercise #2:** A seed company is trying to determine the effect of synthetic nutrients versus organic nutrients on the growth rate of corn plants. They select 40 seeds and randomly distribute the seeds to two groups of 20. The seeds in Group 1 are given the organic nutrients and the seeds in Group 2 are given the synthetic nutrients. After three weeks each plant's growth is measured in centimeters.

Group 1 (Organic): 6, 8, 10, 12, 12, 12, 13, 13, 14, 14, 16, 16, 17, 17, 18, 18, 20, 20, 22, 25

Group 2 (Synthetic): 9, 11, 12, 12, 15, 15, 15, 16, 17, 18, 18, 18, 19, 19, 19, 21, 21, 22, 24, 28

Enter these two lists in your calculator. State the mean of each. Then, create a box plot for each using the grid below. You may want to summarize the information you need under each heading.

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
</tr>
</tbody>
</table>
**Exercise #3:** Let's look at the descriptive statistics we have so far: the sample means and the box-plot. What does this data suggest? Does it give a clear indication that one treatment resulted in greater plant growth?

There are very sophisticated techniques to probabilistically determine what portion of the variability in these two data sets is due to natural causes and what is induced. But, we can run a simple simulation which can give us a very good sense. Consider just the question of the difference in the sample means. The program MEANCOMP will take our two groups of data and randomly scramble them up into two new groups. It will do that over and over again and calculate the difference in the means of the groups.

**Exercise #4:** If $\bar{x}_1$ represents the mean of Group 1 and $\bar{x}_2$ represents the mean of Group 2, do the following.

(a) Find the observed difference in the sample means:

$$\bar{x}_2 - \bar{x}_1 =$$

(b) Run the program MEANCOMP with 100 simulations. Use your calculator to create a frequency histogram on the axes below for the sample mean differences. Point out where on the histogram the observed difference falls.

(c) Look at the data list containing the sample mean differences. Given there are 100 differences, what percent of the differences were at or above the observed difference?

(d) How confident are you about the observed difference in sample means being due to induced variability and not natural variability? Justify.
THE DIFFERENCE IN SAMPLE MEANS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. In an experiment, two main types of variability are introduced. Explain how both affect the results of the experiment. Give examples to support your descriptions.

   Induced Variability  Natural Variability

2. A simulator takes the data from the various treatments, randomly scrambles them together to create groups that contain mixed treatments. Explain how this helps quantify the question of natural variability.

3. Researchers in a sleep lab at a college decide to see how a night of no sleep affected the ability of volunteers to answer 50 addition problems in a minute time span. Thirty volunteers were randomly assigned to two groups. Group 1 was not allowed sleep and Group 2 slept normally. Their results, in terms of questions answered out of 50, are given below. As in the lesson, find the sample means and graph a box plot for each.

   Group 1: 11, 14, 16, 17, 23, 25, 25, 27, 30, 31, 33, 34, 34, 36, 38
   Group 2: 18, 22, 24, 25, 30, 30, 32, 33, 34, 34, 36, 37, 42, 44, 48

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>40</td>
<td>50</td>
</tr>
</tbody>
</table>
4. Does it appear that getting sleep helps in the ability to answer addition problems? What descriptive statistics can you use to strengthen your argument?

5. Is it true that a person who gets sleep will always answer more addition problems than a person who has not gotten any sleep? Support your answer from the experimental results.

6. Run the program MEANCOMP with 100 simulations on these two data sets. Using your calculator, create a frequency histogram for the sample mean differences on the axes below. Mark on the distribution where the observed difference in the sample means lies.

   Observed Difference:

   \[ \bar{x}_2 - \bar{x}_1 = \]

7. What percent of the simulated differences were greater than or equal to our observed differences? Show your calculation below.

8. Can we **confidently** conclude that the variability in sample means is due to the treatment or due to **natural variability**? Support your argument using the distribution above and your answer to 7.
**LINEAR REGRESSION AND LINES OF BEST FIT**
**COMMON CORE ALGEBRA II**

Oftentimes in science, a mathematical relationship between two variables is desired for predictive purposes. In the real world, the relationship between two variables is not always a perfect one, thus we often look for the “best” curve that can fit the data. Today we will review how to do this with a linear function.

**Exercise #1**: A pediatrician would like to determine the relationship between infant female weights versus age. The pediatrician studies 100 newborn girls and finds their average weight at the end of 3 month intervals. The data is shown below and graphed on the scatter plot.

<table>
<thead>
<tr>
<th>Age (months)</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Weight (pounds)</td>
<td>7.2</td>
<td>12.2</td>
<td>15.1</td>
<td>19.4</td>
<td>21.5</td>
<td>26.3</td>
</tr>
</tbody>
</table>

(a) Using a ruler, draw a line that you think best fits this data. As a general guideline, try to draw it such that there are as many data points above the line as below it.

(b) By picking two points that are on the line (not necessarily data points), determine the equation of your best fit line. Round your coefficients to the nearest tenth.

(c) Using the linear regression command on your calculator, find the equation of the best fit line.

(d) Use your calculator to determine the linear correlation coefficient. Round to the nearest thousandth. How can you interpret this value in terms of the variation in weight due to age?
**Exercise #2:** Using the equation that your calculator produced in Exercise #1, predict the weight of a baby girl after 10 months. Round your answer to the nearest tenth of a pound.

The use of a model to predict outputs when the input is within the range of the known data is called **interpolation**. Interpolation tends to be fairly accurate.

**Exercise #3:** Using the equation that your calculator produced in Exercise #1, predict the weight of a baby girl after 2 years. Round your answer to the nearest tenth of a pound.

The use of a model to predict outputs when the input is outside of the range of the known input data is called **extrapolation**. Models are most helpful when they can be used to extrapolate, but tend to be less accurate.

**Exercise #4:** Biologists are trying to create a least-squares regression equation (another name for best fit line) relating the length of steelhead salmon to their weight. Seven salmon were measured and weighed with the data given below.

<table>
<thead>
<tr>
<th>Length (inches)</th>
<th>22</th>
<th>24</th>
<th>28</th>
<th>34</th>
<th>39</th>
<th>42</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (pounds)</td>
<td>3.43</td>
<td>4.46</td>
<td>7.08</td>
<td>14.21</td>
<td>22.19</td>
<td>31.22</td>
<td>35.67</td>
</tr>
</tbody>
</table>

(a) Determine the least-squares regression equation, in the form \( y = ax + b \), for this data. Round all coefficients to the nearest hundredth.

(b) Using your equation from part (a), determine the expected weight of a salmon that is 30 inches long.

(c) Using your equation from part (a), determine the expected weight of a salmon that is 52 inches long.

(d) In which part, (b) or (c), did you use interpolation and in which part did you use extrapolation? Explain.
**LINEAR REGRESSION AND LINES OF BEST FIT**

**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

1. Which of the following linear equations would best fit the data set shown below?

   (1) \( y = 2.4x + 18.7 \)  
   (2) \( y = -0.8x + 18.1 \)  
   (3) \( y = -1.6x + 27.2 \)  
   (4) \( y = 1.9x - 15.6 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>5</th>
<th>9</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>26</td>
<td>17</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>

2. A scatter plot is shown below. Which of the following *could* be the equation of the best fit line for the data set?

   (1) \( y = 1.8x - 3.2 \)  
   (2) \( y = -3.5x - 12.4 \)  
   (3) \( y = -2.9x + 8.3 \)  
   (4) \( y = 6.5x + 3.9 \)

3. A line of best fit was created for a data set that only included values of \( x \) on the interval \( 12 \leq x \leq 52 \). For which of the following values of \( x \) would using this model represent extrapolation?

   (1) \( x = 26 \)  
   (2) \( x = 50 \)  
   (3) \( x = 14 \)  
   (4) \( x = 6 \)

4. Which of the following is true about the line of best fit for the data set given in roster form below?

   (1) It has a positive slope and negative \( y \)-intercept.  
   (2) It has both a positive slope and \( y \)-intercept.  
   (3) It has both a negative slope and \( y \)-intercept.  
   (4) It has a negative slope and positive \( y \)-intercept.

**APPLICATIONS**

5. An agronomist is studying the height of a corn plant as a function of the number of days since the corn germinated (appeared above the ground). Based on the following data, use your calculator to determine the best fit line in \( y = ax + b \) form. Round all coefficients to the nearest tenth.

<table>
<thead>
<tr>
<th>Time, ( x ) (days)</th>
<th>3</th>
<th>8</th>
<th>12</th>
<th>20</th>
<th>28</th>
<th>32</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height, ( y ) (inches)</td>
<td>2.5</td>
<td>4.5</td>
<td>6.2</td>
<td>9.3</td>
<td>12.9</td>
<td>14.4</td>
<td>16.8</td>
</tr>
</tbody>
</table>
6. Heavier cars typically get worse gas mileage (their miles per gallon) than lighter cars. The table below gives the weight versus the highway gas mileage for seven vehicles.

<table>
<thead>
<tr>
<th>Vehicle Weight (thousands of pounds)</th>
<th>2.5</th>
<th>2.9</th>
<th>3.1</th>
<th>3.0</th>
<th>4.2</th>
<th>6.6</th>
<th>3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas Mileage (miles per gallon)</td>
<td>34</td>
<td>36</td>
<td>31</td>
<td>29</td>
<td>23</td>
<td>12</td>
<td>26</td>
</tr>
</tbody>
</table>

(a) Determine the best fit linear equation, in \( y = ax + b \) form, for this data set. Round all coefficients to the nearest tenth.

(b) Using your model from part (a), determine the gas mileage, to the nearest mile per gallon, for a vehicle that weighs 3500 pounds.

(c) Is the prediction you made in (b) and example of interpolation or extrapolation? Explain.

(d) What does the slope of this model represent? Think about your answer to part (c).

7. The superintendent of the Clarksville Central School District is attempting to predict the growth in student population in the coming years. The table below gives the population for her district for selected years.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>District Population</td>
<td>3520</td>
<td>3605</td>
<td>3771</td>
<td>3860</td>
<td>4135</td>
<td>4285</td>
</tr>
</tbody>
</table>

(a) Find the equation for the line of best fit, in \( y = ax + b \) form, where \( x \) represents the years since 1990 and \( y \) represents the district’s population. Round all coefficients to the nearest \( \text{hundredth} \).

(b) Use your model from part (a) to predict the district’s population in the year 2020. Round your answer to the nearest whole number.

(c) What are the units of the slope of this linear model?

(d) What does the slope of this model represent? Why is it negative?
OTHER TYPES OF CORRELATION
COMMON CORE ALGEBRA II

Just as we fit data with a linear model we can also fit with all sorts of other mathematical models, depending on the context of the situation. In this lesson we will examine exponential regression and sinusoidal regression. Exponential regression is review from Common Core Algebra I, so we will start with that.

Exercise #1: The population of Jamestown has been recorded for selected years since 2000. The table below gives these populations.

<table>
<thead>
<tr>
<th>Year</th>
<th>2002</th>
<th>2004</th>
<th>2005</th>
<th>2007</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>5564</td>
<td>6121</td>
<td>6300</td>
<td>6812</td>
<td>7422</td>
</tr>
</tbody>
</table>

(a) Using your calculator, determine a best fit exponential equation, of the form \( y = a \cdot b^x \), where \( x \) represents the number of years since 2000 and \( y \) represents the population. Round \( a \) to the nearest integer and \( b \) to the nearest thousandth.

(b) Sketch a graph of the exponential function for the years 2000 to 2050. Label your window and your \( y \)-intercept.

(c) By what percent does your exponential model predict the population is increasing per year? Explain.

(d) Algebraically determine the number of years, to the nearest year, for the population to reach 20 thousand.

Exercise #2: Which of the following scatter plots would be best fit with an exponential equation?

(1)  
(2)  
(3)  
(4)  

Common Core Algebra II, Unit #13 – Statistics – Lesson #9  
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Sinusoidal, or trigonometric, regression is much more complicated than either linear or exponential. It should be used in situations that appear periodic in nature.

**Exercise #3:** The temperature of a chemical reaction changes during the reaction. The temperature was measured every two minutes and the data is shown in the table below.

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>35.7</td>
<td>38.9</td>
<td>41.6</td>
<td>42.3</td>
<td>40.8</td>
<td>38.4</td>
<td>36.1</td>
<td>34.2</td>
<td>35.9</td>
<td>39.1</td>
<td>41</td>
</tr>
</tbody>
</table>

(a) Why does it seem like this data might be periodic? Create a quick scatter plot using your calculator to verify.

(b) Use your calculator to do a sine regression in the form \( y = a \sin(bx + c) + d \). Round all parameters to the nearest tenth. Graph along with your data to informally assess the fit of the curve.

(c) According to this model, what is the range in temperatures the chemical reaction will include?

(d) According to this model, what is the time it takes for the reaction to complete one full cycle?

Graphing calculators vary. Many will require that if the period of the sinusoidal function is unknown then the data must have inputs that are separated by equal amounts (equal steps between \( x \)-values). On the other hand, there are many periodic phenomena that we want to model whose periods are known. In this case, we can enter data at irregular input intervals.

**Exercise #4:** The maximum amount of daylight that hits a spot on Earth is a function of the day of the year. Taking \( x = 0 \) to be January 1st, daylight, in hours, was measured for 12 different days. The measurement was the number of possible hours of sun from sunrise to sunset.

<table>
<thead>
<tr>
<th>Day</th>
<th>0</th>
<th>34</th>
<th>68</th>
<th>98</th>
<th>118</th>
<th>134</th>
<th>171</th>
<th>203</th>
<th>274</th>
<th>321</th>
<th>346</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daylight Hours</td>
<td>9.0</td>
<td>9.9</td>
<td>11.5</td>
<td>13.1</td>
<td>14.0</td>
<td>14.6</td>
<td>15.2</td>
<td>14.8</td>
<td>13.1</td>
<td>11.5</td>
<td>9.5</td>
</tr>
</tbody>
</table>

(a) What is the natural period of this data set?

(b) Use your calculator with the period from (a) to find an equation of the form \( y = a \sin(bx + c) + d \) that fits this data, then examine the graph of the equation on the scatter plot. How good is the fit?

(c) What is the maximum amount of daylight hours predicted by the model? Show your calculation.
OTHER TYPES OF CORRELATION
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. Rabbits were accidentally introduced to an island where their population is growing rapidly. Biologists studying the rabbits have periodically recorded their population since they were introduced to the island. The data they took is shown below.

<table>
<thead>
<tr>
<th>Years Since Introduction, x</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population of Rabbits, y</td>
<td>75</td>
<td>100</td>
<td>112</td>
<td>205</td>
<td>290</td>
</tr>
</tbody>
</table>

(a) Determine an exponential regression equation, in the form \( y = a \cdot b^x \), that models this data. Round \( a \) to the tenth and \( b \) to the hundredth.

(b) Sketch a graph of the rabbit population below on the axes provided for \( 0 \leq x \leq 20 \). Label your graphing window and your \( y \)-intercept.

(c) Based on your model in part (a), by what percent is the rabbit population growing each year?

(d) Graphically determine, to the nearest tenth of a year, when the rabbit population will reach 350.

2. The infiltration rate of a soil is the number of inches or water per hour it can absorb. Hydrologists studied one particular soil and found its infiltration rate decreases exponentially as a rainfall continues.

<table>
<thead>
<tr>
<th>Time, ( t ) (hours)</th>
<th>0</th>
<th>1.5</th>
<th>3.0</th>
<th>4.5</th>
<th>6.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infiltration Rate, ( I ) (inches per hour)</td>
<td>5.3</td>
<td>3.1</td>
<td>2.4</td>
<td>1.6</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Create an exponential model that best fits this data set. Round parameters to the nearest hundredth. Use your model to algebraically determine the time until the rate reaches 0.25 inches per hour. Round your answer to the nearest tenth of an hour. Use a logarithm in the process of your algebraic solution.
3. The soil's temperature beneath the ground varies in a periodic manner. A temperature probe was left 3 feet underground and recorded the temperature as a function of the number of days since January 1st \((x = 0)\). The temperatures for 14 days throughout the year are shown below.

<table>
<thead>
<tr>
<th>Day</th>
<th>5</th>
<th>36</th>
<th>57</th>
<th>94</th>
<th>127</th>
<th>153</th>
<th>192</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°F)</td>
<td>41</td>
<td>37</td>
<td>36</td>
<td>40</td>
<td>48</td>
<td>64</td>
<td>68</td>
</tr>
<tr>
<td>Day</td>
<td>226</td>
<td>241</td>
<td>262</td>
<td>289</td>
<td>305</td>
<td>337</td>
<td>356</td>
</tr>
<tr>
<td>Temp (°F)</td>
<td>66</td>
<td>61</td>
<td>58</td>
<td>49</td>
<td>44</td>
<td>42</td>
<td>40</td>
</tr>
</tbody>
</table>

(a) Find a best fit sinusoidal function for this data set in the form \(y = a \sin (bx + c) + d\). Round all parameters to the nearest \(\text{hundredth}\). Recall that some calculators require that you input the period on this correlation (365 days).

(b) Based on your model from (a) what are the highest and lowest temperature reached in the soil?

(c) What is the average soil temperature?

(d) If the root of a particular plant species will only thrive when the soil temperature is above 50 °F, graphically determine the interval of days over which the plant will thrive.

4. The rise and fall of the tides at a beach is recorded at regular intervals. Their period is almost 24 hours, but not exactly. The depth of a tidal marsh was measured over 3-hour time interval and the data is shown below.

<table>
<thead>
<tr>
<th>Hours (since midnight)</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth (ft)</td>
<td>5.5</td>
<td>8.0</td>
<td>10.5</td>
<td>11.7</td>
<td>10.8</td>
<td>8.4</td>
<td>5.8</td>
<td>4.3</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Find a sinusoidal model for this data using your calculator. Place it in \(y = a \sin (bx + c) + d\) form. Round all coefficients to the nearest \(\text{thousandth}\) (3 decimal places).

According to your model, what is the period of the tides in hours? Recall that \(b \cdot P = 2\pi\).